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ABSTRACT

A study was conducted to examine the learning of mathematical concepts by English dominant and Spanish dominant primary school children. The concepts considered were number, numeration, and computational procedures. Using the case study method, a three step procedure was designed to study individual math objectives: (1) objectives were first identified in the curriculum; (2) these objectives were then included in a teacher survey for ratings on item difficulty as perceived by the teachers; and (3) students were observed while learning the identified math objectives. Among the findings of the study were: (1) field independence tends to be associated with achievement in mathematics, while field sensitivity tends to be associated with lower levels of achievement; (2) field independent students tend to perform better on the precise thinking exercises, demonstrating a higher ability to use and apply rules; (3) the ages of children within a grade do not seem to be related to achievement; (4) children's language skills have not completely developed by the first grade, and there are specific usages that occur in math teaching which cause children difficulty; and (5) bilingual Hispanic students tend to perform at higher levels than do English or Spanish monolingual students. (Author/APM)

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A Final Report
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By

Dr. Jack A. Coffland
Dr. Gilbert J. Cuevas

School of Education
and Allied Professions
University of Miami
Coral Gables, Florida

July, 1979

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## Chapter One

## OVERVIEW

### General Description of Purpose

Several basic assumptions were made at the inception of the study which set the basic theoretical foundation for subsequent research procedures. These assumptions were as follows: first, "two interrelated strands of instruction in arithmetic appear" (NIE, 1977) from the earliest grades. These two strands, number and numeration, are different and distinct, but definitely interrelated. Problems in development of number concepts may stem from either strand, or may be the result of complex interactions between the development of the two concepts. As a result, the concepts of number and numeration were examined, especially as they come together in numerical operations.

Secondly, "since elementary mathematics is to a large extent a cumulative subject, it is apparent that children without adequate concepts of the decimal numeration system are at an increasing disadvantage as they encounter more sophisticated ideas in arithmetic" (NIE, 1977). Because of this assumption, it was determined that an investigation into children's early learning of number and numeration topics might help explain some subsequent problems which they encounter.

Finally, there is "some evidence that certain content areas in mathematics are frequently more difficult for children from cultural, social and linguistic minorities" (NIE, 1977). As a result, the project focused attention on the development of math concepts in two different groups: English-speaking students and Spanish-speaking students. No

effort was made to compare the groups; rather, emphasis was placed upon the development of concepts in both samples.

The general purpose, then, was to study the development of the concepts of number and numeration in primary age children from two linguistic groups. These conceptual strands were traced over a two year period during the early stages of each child's involvement with formal instruction in arithmetic. Through such a process, it was hypothesized that research questions might be generated about young children's learning of math concepts.

## Statement of the Problem

Specifically, the investigation dealt with five goals. They were:

1. Increased knowledge about the nature and development of children's ideas of number and numeration in the primary grades (usually thought of as kindergarten through the third or fourth grades).
2. An understanding of how children use concepts of number and numeration in early mathematical tasks, including tasks that are currently part of the standard primary mathematics program.
3. A better understanding of the nature of the difficulties that some children have with these concepts and tasks.
4. Prototype diagnostic instruments that can be individually administered to children in the primary grades to assess their knowledge of number and numeration, independently of any particular text or program.
5. A set of hypotheses about instructional materials or techniques that might be particularly effective in teaching concepts of

number and numeration and their use in mathematical tasks to children who have difficulties in grasping the fundamental notions.

## General Description of Procedures

Several steps were outlined for completion of the research project. First, it was decided that a survey of the existing mathematics curriculum be completed in order to determine what specific objectives are being taught under the general headings of number and numeration. The curriculum chosen for this survey was the Dade County Systems Math Program, as used in Dade County, Florida. This program includes objectives from all commercial math programs, organized into a logical, sequential program of twenty-eight levels. It begins in kindergarten and goes through the sixth grade. Such a curriculum was chosen for two reasons: it lists each objective for the mathematics program, and it is independent of any given textbook series.

Secondly, it was decided that the number and numeration objectives taught to primary age children would be organized into a questionnaire so that teachers could rate the difficulty of the various objectives, as they perceived that difficulty. It was felt that teachers represent a data base which is often overlooked in doing research; teachers have a long, experiential record of teaching the objectives to various age groups of children. This experience is invaluable in determining which objectives are difficult, in locating possible reasons why they are difficult, and in explaining some problems children have in dealing with the objectives.

Finally, the researchers followed children through two years of their elementary education, observing, interviewing, tutoring, and testing the children to ascertain the difficulties and successes they had in dealing with the concepts of number and numeration. Special attention, especially in the testing program, was given to those objectives identified as being "difficult" by a majority of the teachers responding to items in the teacher survey. As one goal of the study was to raise hypotheses, it was decided to treat all observational data as case study research data.

## General Areas of Concern

The purpose of the study, during its two-year duration, was to gather data related to the learning of mathematical concepts by primary grade students who are Spanish or English language dominant. The efforts of the investigators in the study and analysis of the students' behaviors were concentrated in the following areas:

1. The concept of number: This strand was related to the Piagetian concept of number, which includes amount, the ability to conserve amount over physical changes, the concept of reversibility, set equality and inequality, and other such concepts as discussed by Piaget.
2. The concept of numeration: Emphasis here was upon the use of the place value system; especially the ways of teaching place value and the rules for organizing the place value system.
3. The basic operations of addition, subtraction, multiplication and division: The use of manipulatives and rule governed

behavior were studied as they related to the teaching/learning process for basic operations. Each operation was examined in detail, though the operations of multiplication and division were not carried through to their conclusion, as only primary grade concepts were studied.

4. The language involved in mathematics concepts: Language terms and patterns used to describe mathematics concepts were examined. It was assumed that if the child did not understand the terms or phrases used to describe a concept, he/she would not be able to understand the concept itself.

These topics comprised the curricular areas for study in this research project. They defined the topics relevant to the general strands of number and numeration which were required in the study.

## Summary

A summary definition of the study would include the following main points:

1. The study focused on primary grade children from two language dominant groups - Spanish and English.
2. Math concepts dealing with number, numeration and computational algorithms were observed. Language used to describe these concepts was also observed.
3. A three step procedure was followed to investigate individual objectives. Objectives were first identified in the curriculum; they were then included in a teacher survey for ratings on item difficulty as perceived by the teachers. Finally, children were observed while learning the identified objectives.

4. A case study approach was utilized for all observational data gathering purposes.

Following sections of this report will detail procedures used and findings obtained.

## Chapter Two

## METHODOLOGY

Three general topics will be discussed in this chapter. First, the content analysis procedures used in isolating objectives taught under the headings of number and numeration will be described. Secondly, the development of the teacher survey questionnaire and the procedures used in administering that questionnaire will be covered. Finally, a description and rationale of the case study procedures used in working with English-dominant and Spanish-dominant primary grade children will be described. The sample used will be identified and described, then the instruments used in data gathering procedures will be identified and described.

### Content Analysis

The researchers' initial step was to identify those objectives being taught under the headings of number, numeration, and computational procedures. In order to accomplish this goal, the existing math curriculum for Dade County, Florida, was analyzed to determine the specific objectives taught. The objectives were then classified under the headings desired or identified as being outside the scope of this study.

The Dade County math program, called the Dade County Systems Math Program, was used for the content analysis for two reasons. First, the content of the program is not dependent upon an individual textbook series or math program. The systems program lists all objectives which should be taught, then recommends sources for teaching activities, drill pages, or concept development materials from all existing textbooks and

auxiliary materials. The objectives, therefore, were general to all math programs.

Secondly, the objectives in the systems program were those actually taught to primary age children. This allowed the researchers to observe children attempting to learn the stated objectives. Such observations made it possible to observe the difficulties children had with explicitly stated mathematics behaviors.

Objectives were classified as being in the following general areas:

1. Number
2. Numeration and Place Value
3. Addition
4. Subtraction
5. Multiplication
6. Division

Objectives dealing with such topics as geometry, measurement, time, money or other such variables were outside the scope of this study and were omitted from the content analysis.

The classification lists were then presented to a mathematician for review. Her suggestions were utlized in making minor revisions in the classification lists; objectives identified by this individual as being possible considerations for more than one category were examined in detail. Actual test items for the questioned objectives were consulted in order to ascertain the specific purpose of the objective. Some objectives were reclassified according to this procedure.

who taught math responded to the survey; in some schools with high ability students, the teachers of fifth and sixth grade classes did not have any students working at the lower levels and consequently did not respond to the questionnaire.

Teachers were asked to fill out those sections of the questionnaire dealing with the levels stressed in their own classroom. No teacher completed more than four levels, while the majority completed two or three levels. The actual breakdown of respondents by levels of the questionnaire is shown in Table I.

Table I

Distribution of Respondents
by Questionnaire Levels

| Systems Level | N of Respondents | % of Respondents |
|---|---|---|
| I | 30 | 14.4% |
| II | 42 | 20.2% |
| III | 62 | 29.8% |
| IV | 56 | 26.9% |
| V | 51 | 24.5% |
| VI | 52 | 25.0% |
| VII | 50 | 24.0% |
| VIII | 47 | 22.6% |
| IX | 29 | 13.9% |
| X | 31 | 14.9% |
| XI | 25 | 12.0% |
| XII | 25 | 12.0% |
| XIII | 27 | 13.0% |
| XIV | 26 | 12.5% |
| XV | 23 | 11.1% |

N = 208

As can be seen in Table I, the greatest number of teachers responded to Levels II through VIII of the program; these levels comprised the major portion of the curriculum for grades one and two. The number of respondents decreases towards the upper levels, levels which only the most gifted first or second grade student would reach.

In this manner the questionnaires were distributed by the researchers and completed by the faculty members of the thirteen selected schools. All who took part in the study completed at least one level of the questionnaire; there were no "non-returned" questionnaires.

Data were analyzed in three steps. First, a general description of the results was made. Secondly, responses were broken down between the teachers of English-dominant children and teachers of Spanish-dominant children. A Chi-square statistic was computed to test the significance of differences between the responses of the two groups at the .05 level. Finally, a Cramer's V statistic was computed to test the strength of the relationship identified by the Chi-square test.

## Classroom Visitations

The major portion of the research commitment involved visitations in the classrooms selected to take part in the study. A case study approach towards the students involved was followed for the purpose of making developmental inferences. Data were collected by:

1. Observation of children's interaction in the classroom and their performance on assignments. The investigators recorded the behavior of students and teachers as topics in number, numeration and the basic operations were presented in the class.

Observations focused on the manner and the language the teacher used to teach the lesson, and the students' understanding of the material was noted. Children's papers were observed and sometimes collected to illustrate specific points.

2. Tutoring students working on specified content area objectives. Teachers referred students to the investigators for tutoring sessions on the basis of the pupil's lack of understanding of a particular topic. The investigator's role was to explain the concept or procedure to the student while noting the approaches the child made to understand the material.
3. Interviews of students in order to describe thinking patterns used in problem solving or algorithms. The interviews were structured around mathematical exercises, such as completing the problem 3 + __ = 5, or other such topics as word problems, Piagetian tasks, or conceptual activities. The students' performance and comments were recorded.
4. Testing and recording of the children's use of mathematics concepts, ideas, vocabulary, or other relevant items. The purpose of this phase of the classroom visitations was to inquire into specific abilities of the students in the areas under consideration.

Such a program of data gathering in the classroom situation has been classified as "naturalistic" in essence. In such an inquiry, the researcher commences with a set of beliefs, frames of references or convictions; these are followed by observations of behavior during a series

of occasions. According to Butler (1963) it is essential to take into account "... not only the sequence of occasions, but ... the relationships among the behaviors over the whole set of occasions without regard to the structure of each (event.)"

The primary function of the research methodology is to generate hypotheses concerning the underlying structure of behaviors; it is not to test hypotheses. Butler (1963), in summarizing the nature of exploratory research, stated that it:

> "...starts and ends with personal creation. Usually personal creation ... should be confined to the beginning of the investigation ... and to the end when the routines of analysis have provided the formal results upon which personal creativity should operate."

Within the framework of a case study approach, then, several specific research considerations will be discussed in subsequent sections of this chapter. The sample selected will be described. The instruments given as tests, and the Piaget tasks given in test-like situations will also be described.

## Sample

During the first year of the study, students in two first grade classrooms at two different elementary schools were chosen as the sample for the study. The first group, from "School A," a Dade County, Florida public elementary school, was chosen because of the predominant Spanish-speaking population attending that school. Twenty-five children in a first grade classroom were selected to participate in the study.

The second group, from "School B," another Dade County, Florida public elementary school, was chosen because students at this school were predominantly English-speaking. Twenty-six children in a first grade classroom were included in this group.

As can be seen in Table II, the breakdown of students by age was approximately the same for both schools. All but two students were six years of age, as is the norm for the first grade. The breakdown by sex indicated that the two school samples were different. School A had a predominantly male sample, while School B had a predominantly female sample. Finally, it can be seen that School A had a predominantly Spanish-speaking group, while School B was predominantly English-speaking. School A had only three students in the sample whose dominant language was English, while School B had only one student whose dominant language was Spanish.

The teachers at both schools were female, with the instructor at School A having more years of experience than the one at School B. Both teachers were competent professionals and were highly rated in professional activities by their principals.

Table II

Description of First Year Sample
By School, Sex, Age and Dominant Language

| SCHOOL/LANGUAGE | AGE GROUP | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| | 5.5 - 5.9 | | 6.0 - 6.4 | | 6.5 - 6.9 | | 7.0 + | |
| | Male | Female | Male | Female | Male | Female | Male | Female |
| School A (n = 25) | | | | | | | | |
| Spanish (n = 22) | - | - | 6(24%) | 4(16%) | 8(32%) | 3(12%) | 1(4%) | - |
| English (n = 3) | - | - | - | 2(8%) | - | 1(4%) | - | - |
| School B (n = 26) | | | | | | | | |
| Spanish (n = 1) | - | - | - | - | - | 1(3.85%) | - | - |
| English (n = 25) | - | 1(3.85%) | 2(7.69%) | 12(46.15%) | 5(19.23%) | 5(19.23%) | - | - |

During the second year of the study, the majority of students were followed into the second grade year. In addition, two new first grade classrooms were added to the sample selected for the study. A Spanish-speaking classroom was selected at "School C," another Dade County, Florida public elementary school. This group was a higher achieving group than the first year's Spanish-speaking sample. The first-year, bilingual group was chosen expressly to view the problems children have with learning mathematics. The second-year, bilingual group was chosen to see what successes bilingual children have, and how language is related to success in mathematics achievement. The two groups provided an indication of the broad spectrum of mathematics achievement present in Hispanic children.

A second, first-grade group was also chosen at "School B." This group was added in an effort to further investigate children's learning of concepts identified as important during the first year of the study.

Table III shows the framework of the student samples used in relation to the time involvement in the study.

Table III

Subject Time Involvement

| SCHOOL | 1977 - 1978 | 1978 - 1979 |
|---|---|---|
| School A | First Grade | Second Grade |
| School B | First Grade | Second Grade |
| School B | | First Grade |
| School C | | First Grade |

The breakdown of the second group of students' age, sex, language dominance and school is shown in Table IV. As can be seen, the sample selected from School C was predominantly Spanish-speaking, with approximately equal divisions between male and female students. On the other hand, School B students were predominantly English-speaking, again with an equal division between male and female students.

The teachers for the second-year, first grade children were both female. The School B instructor was the same teacher that had the students during the first year of the study. She was a bilingual female, rated as excellent by her principal. The School C, first grade teacher was a monolingual female, again with more years of experience than the School B teacher, and was rated as an excellent teacher by the school principal.

The teachers for the second-year, second grade children were also female. At School B, the instructor was again the same individual. Her schedule included second grade math instruction in the morning and first grade math instruction in the afternoon. At School A, the second grade children were divided between two classrooms. Both teachers were monolingual, and both were rated excellent by the principal.

Table IV

Description of Second Year, First Grade Sample
By School, Sex, Age and Dominant Language

| SCHOOL/LANGUAGE | AGE GROUP | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| | 5.5 - 5.9 | | 6.0 - 6.4 | | 6.5 - 6.9 | | 7.0 + | |
| | Male | Female | Male | Female | Male | Female | Male | Female |
| School C (n = 25) | | | | | | | | |
| Spanish (n = 20) | 2(8%) | 2(8%) | 4(16%) | 3(12%) | 2(8%) | 7(28%) | - | - |
| English (n = 5) | - | - | 1(4%) | - | 1(4%) | 3(12%) | - | - |
| School B (n = 29) | | | | | | | | |
| Spanish (n = 1) | 1(3.5%) | - | - | - | - | - | - | - |
| English (n = 28) | 6(20.7%) | 2(6.9%) | 8(27.6%) | 6(20.7%) | 1(3.5%) | 4(13.8) | - | 1(3.5%) |

Instrumentation

A large number of tests and Piagetian tasks were given to the children over the two-year period of the study. This was in line with the case study design, in which the major purpose was to pose hypotheses, rather than to formally test them with a research design. The tests given are detailed in the subsequent pages of this report.

A. Metropolitan Readiness Test: The MRT (Level II, Form P) was administered to each student in the sample at the beginning of his/her first grade year. It was given in order to obtain information about each pupil's levels of development in pre-reading and pre-mathematics skills. As stated in the Teacher's Manual of the 1976 edition of the MRT, the test is designed to assess the range of pre-reading and mathematical skills present in the child at the end of kindergarten and at the beginning of Grade 1. The test consists of the following sections:

| | |
|---|---|
| Auditory Memory: | This area assesses the child's immediate recall of initial consonant sounds and of sound/letter correspondence. |
| Visual Discrimination: | This area assesses the skills of visual matching and finding patterns. |
| Language Skills: | This area encompasses school language and listening skills, using standard American English. |

Quantitative Skills: This area covers quantitative concepts, such as number-numeral correspondence, conservation of amount, part-whole relationships and quantitative operations.

Data obtained were utilized by the researchers to provide a context in which to relate the subsequent successes and/or failures experienced by pupils as they learned the concepts and operations involved in number, numeration, and computational procedures.

B. Children's Embedded Figures Test: Cognitive style was identified in the research proposal as a variable which might influence achievement in mathematics. The CEFT is a perceptual test; and as stated in the test manual: "The subject's task on each trial is to locate a previously seen simple figure within a larger complex figure which has been so originated as to obscure or embed the sought after simple figure." High ability on the test (norm = 6.8 for 5 - 6 year old males, SD = 3.8; 7.4 for 5 - 6 year old females, SD = 4.2) classifies a student as being field independent; low ability on the test classifies a student as being field dependent.

The CEFT was given to all of the students in the sample at the beginning of their first grade year.

C. Stanford Achievement Test: As part of the district wide testing program in the public schools, all students take the Stanford Achievement Test during each of their school years, unless their English ability is felt to be too low to understand the test. These data were used to support or supplement observational data on the child's general mathematics achievement. The test during school year 1977-78 was given in February

of 1978, the normal test administration date. Due to a district-wide administrative problem, the SAT was given to all students in elementary school at the end of the 1978-79 school year. This administration of the test serves as an excellent concluding measure of the students' mathematical achievement.

D. Dade County Systems Math Program Tests: Teachers administrate a diagnostic/prescriptive program designed to guide instruction and verify children's success with specific objectives. This is part of the normal math teaching procedures for the school district. Systems math results were available for children and served as an indicator of general math achievement. More importantly, the students' records on program objectives displayed the day by day difficulties children had with mathematics objectives.

E. Piaget Tasks: In an effort to obtain data on the child's cognitive development, tasks developed and publicized by Piaget were used to look at overall development and development of concepts in a specific area.

Four of the tests used to assess the Piaget level of cognitive development were conservation tests. They included the following:

1. Conservation of Length: The child was introduced to two lines of equal length. After the equality of length had been established, one line was rearranged so that it was pushed somewhat to the right or to the left of the other line, but was still paralleling it. The child was asked if the two lines were the same or different in regard to length. After this judgment

the child was questioned about his/her reasons for making the judgment. A second judgment was obtained when one of the lines was arranged in a zig-zag pattern and hence its end coincided with a point two-thirds from the end of the reference line.

2. Conservation of Area: This was tested using two equal green boards pretended to be fields of grass. Two animals were placed on the fields and the child was required to judge whether one animal had more or less grass to eat or did both have the same amount. After the initial equality of the areas had been established, "barns" were built on both fields (actually, wooden blocks) to cover some of the grass in both fields. The barns on one field were built close together and on the other field spread throughout the field. After the child made a judgment with five barns in each field, he was asked the same question with ten barns in each field. After each judgment the child was asked for an explanation of his/her reasoning.
3. Conservation of Volume: This task required the child to make three judgments. After the equality of the amounts of liquid in two glasses of equal size and shape had been established, the liquid in one of the glasses was poured into a taller, more slender glass. The child was asked if there was still the same amount of liquid in the two glasses or if one had more than the other. After the liquid was returned to the original glass and equality here was reaffirmed, the liquid was poured into a wider container and eventually into a set of four small glasses. Each time the child was asked to explain his/her judgments of equality.

4. Conservation of Mass: For this task two pieces of play-dough of equal size and shape were used. After their equality had been established, one ball of dough was shaped like a hot dog and the child's judgment was requested. The hot dog was shaped again into a ball, then it was broken into four little balls and the child was asked to make a judgment again about the amount of dough in the original ball and the four little ones. Again the child was required to explain his/her judgment.

Whenever a child made the correct judgment, exhibiting conservation of any of the four types, he/she was asked to prove that the judgment was indeed correct, preferably by manipulating the task's materials.

In addition to the conservation tasks, two additional Piagetian tasks were given to examine children's development with specific tasks. These included:

5. Class-inclusion Task: For this task the child was shown picture-cards with women on four of them and men on two. The child was asked to count the men and the women and to establish the fact that there were more women than men. Then the child was asked if there were more women or more people in the array in front of him. He/she was also asked if there were more people or more men. As in the conservation tasks, the children were asked to justify and explain their judgments. If the child made an incorrect judgment, he/she was asked to count the number of people and make the judgment again to ensure that he/she understood the use of the word people and that

he/she was aware of the number of people involved. The second class-inclusion task was the same but picture-cards of various fruits (five) and cake-like foods (two) were used instead of the men and the women. After counting and pointing to the five fruits and the two cake-like objects, the children were asked if there were more fruits or more things to eat and if there were more baked things or more things to eat.

6. Classification Task: For this task the children were given plastic blocks varying in color, shape and size. There were three colors (red, yellow and blue), two shapes (balls and cubes) and two sizes (large and small); all categories were completely crossed so that there were 12 blocks in all. The child was asked to group them in any number of groups he/she wanted but so that all blocks must be used and the ones placed in a group somehow had to "go together." If the child had difficulty, one way of classifying was demonstrated to him/her.

   After the child grouped the objects, he/she was asked to explain the method of grouping and if there was another way in which he/she could group the blocks. Each child was asked this question until he/she stated that there was no additional way to group them.

All six of these Piagetian tasks were given during the same sitting. Each child was tested individually and the interview lasted 25-30 minutes. These tests were given to students during their first grade year.

F. Math Concept related Piagetian Tasks: A final series of Piagetian tasks were given over the two year period of this study. Each of these tasks was directly related to an objective rated as "difficult" by teachers responding to the teacher survey. These final tasks included twelve different problems for the child to complete. Nine of these tasks were taken directly from the literature on Piaget and mathematics; the remaining three were adaptations of existing tasks from the mathematical literature. These tasks included the following:

1. Addition of Classes Task: The purpose of this activity was to determine when and how the child understands addition as an operation. Addition here was applied to classes of objects, in this case, blocks. It was similar to the class inclusion task, in that the child was required to add two classes in order to form a new class (Copeland, 1974).

   The child was given nine wooden blocks, seven of which were blue and two of which were white, and the following questions were asked:

   (1) What are the blocks made of?

   (2) What color is this (pointing to the blue) and what color is this (pointing to the white)?

   (3) Are there more blue or more wooden blocks?

   (4) If I made a row of wooden blocks and a row of blue blocks, which would be longer? Why?

2. Addition and Subtraction of Numbers: While the first task required that the child combine classes, a variation of the

addition question was: Is the addition of numbers psychologically more difficult than the addition of classes? Such a question can be investigated through the use of various arithmetic combinations (Copeland, 1974).

Given eight blocks, arranged in groups of four, the child was told the following:

> Imagine these are sweets. Today you can have four in the morning and four in the afternoon. Tomorrow you can have one in the morning and seven in the afternoon. (Then move the blocks from the 4 + 4 alignment into a 1 + 7 alignment. Then ask:) Will you have more candy on the second day, or will you have the same on both days?

3. Missing Addend Rod Task: In an effort to gather data on the concept of the missing addend, two tasks were devised for this particular problem type. The first concerned the use of rods. The child was given a large rod; then a short rod was placed on the table, some distance from the large rod. He/she was then asked:

   > What would you have to put with the short rod to make it as long as the large rod?

   A collection of rods of various sizes was then made available to the child. He/she was allowed to move the rods together in order to solve the problem, but was not prompted to do so.

4. Missing Addend Block Task: The second task for the missing addend utilized the phrasing "how many more" instead of "as

long as" in an effort to examine the exact nature of the missing addend situation.

A row of 5 blocks was placed before the child. Another row of two blocks was placed in a second row, near to but not immediately beside the first row. The child was then asked:

> How many more blocks would you need to make the second row have the same number as the first row?

More blocks than necessary were then made available to the child. If the child arrived at an answer which was correct, a second problem (3 + __ = 8) was given. The block rows were placed somewhat closer together, but were arranged in such a way that the three block row occupied the same distance as the first two blocks in the longer row. (In other words, the blocks were not placed in a one-to-one alignment.) The child was then asked:

> How many more blocks would you need in order to make the short row have as many blocks as the long row?

Again, blocks were then made available to the student to use in solving the problem.

5. Seriation Task: The task included the basic Piagetian concept of seriation. It asked that the child order a set of objects using size as the criterion of order. The students were given twelve straws of varying lengths, and two different directions were given to them:

   (1) Arrange the straws in order from smallest to largest.

   (2) Arrange the straws in order from largest to smallest.

6. Place Value Conservation Task: Underwood (1977) tested children with what was termed a "place value conservation task." The basic point for the child was to ascertain if a group of ten objects were viewed in the same manner as the ten objects spread apart. The task consisted of two steps. First the child was shown a group of ten objects plus four left over, then was asked to count them and determine that there were fourteen objects. Then the group was taken apart before the child; the ten sticks were taken from their rubber band and placed individually on the table, along with the four extra sticks. The child was then asked:

   > Are there as many sticks now as there were before, or do we have more or less sticks?

   The task required the child to conserve amount over a physical change; he needed to see that 10 + 4 was the same as 14.

7. Grouping: Another test (Copeland, 1979) of place value concepts examined children's readiness to group objects into groups of ten. This task required that the adult place thirty-four blocks before the child. The child was then told:

   > We would like to group these objects to make them easy to count. Arrange the blocks into some arrangement so I can count them just by looking at the groups.

   The expectation, of course, was that the child would make three groups of ten, with the group of four remaining.

8. The Null Set Task: Copeland (1974) reports resistance on children's part to forming a null or empty set; he goes on to state

that many children seem unable to consider such an intangible concept as a "set of nothing." He reported the following test as a means of examining the child's concept of the null set.

The child is given fifteen cards; arranged in the following manner:

(1) Five triangular cards, three with pictures, two blank.

(2) Five square cards; four with pictures, one blank.

(3) Five circular cards; three with pictures, two blank.

He was then given the following instructions:

(1) "Put these together" or "classify them" in whatever way you choose.

(2) After the child made a classification, then: "Make another classification with only two groups."

The cards were constructed in such a way that the only logical groups that could be formed were a set of picture cards and a set of cards with no pictures, or an empty set.

9. Combinations: Copeland (1979) reports a test by Piaget asking that a child make all possible combinations from the elements of a given set. Such a test has a direct impact on multiplication, as the child is expected to show all possible combinations in a Cartesian cross as a way of defining multiplication. Four combination tasks were given to the child.

First, the child was given a set of four blocks of differing colors and asked to make all possible combinations of two that he could from that set. Since each block was of a different color, color combinations were recorded as a record for the child to see.

He was then given the same set of four blocks and asked to make all possible combinations of three from the set of blocks. Again, his/her choices were recorded by color combinations.

The third task asked the child to take a set of five blocks and make all possible groups of two. The final task was to take the set of five blocks and make all possible groups of three. In all cases the selected combinations were recorded for the child and made available to him for examination.

10. Permutations: A task similar to that of combinations was also given (Copeland, 1979). The child was given a set of four blocks and told to show all possible permutations. (An example was given in place of the term.) This test proved to be too difficult for any child to complete and was dropped from the group of tests.

11. Cartesian Products: Both the combinations and permutations tasks gave one possible way of examining the child's ability to use a logical sequence of behaviors in order to arrive at all possible combinations or permutations from a given set of objects. This served as an oblique way of examining the child's ability to make a Cartesian cross between two sets. But the skill was also tested directly. The test was quite similar to Piaget's combinations test.

Therefore, the child was given two sets of objects; they included:

(1) A set of two triangles, one red and one green.

(2) A set of three squares, one red, one green, and one yellow.

An example was then given of making a group by taking one triangle and one square. The colors were recorded, with the triangle color

always listed first. The child was then asked to make as many additional groups of one triangle and one square as he could.

A second task was then given, with the sizes of the sets increased. A blue triangle and a blue square were added to the original sets, making the sets have respective n's of three and four. The child was then given the same instructions: make as many combinations as possible, but each one must have one triangle and one square.

12. Family Relations Test: Copeland (1979) described a test of Piaget's which attempts to examine the child's knowledge of class inclusion and logical thinking by projecting family relationships in a logical manner.

The test included ten questions over family relationships, especially examining the reciprocal nature of those relationships. The ten questions were:

(1) What is a family?

(2) What is a cousin?

(3) What is an uncle?

(4) What is a brother?

(5) How many brothers do you have?

(6) How many sisters do you have?

(7) How many brothers are there in your family?

(8) How many sisters are there in the family?

(9) How many brothers and sisters altogether?

(10) There are three brothers in a family, David, Denny and Mike.

How many brothers has David?

How many brothers has Denny?

How many brothers has Mike?

(11) Are you a sister or a brother?

Copeland (1979) quotes Piaget as saying: "the inability to grasp the relativity of notions or ideas is one of the principal obstacles to the development of childish reasoning."

G. In addition to the development of cognitive structure tests, language development tests were given. These focused on two general areas: the first area consisted of children's knowledge and use of common language structures, especially those structures which might be used in a mathematics situation or a math class. The second language test focused upon children's knowledge of the terms of mathematics, especially those used in the primary grade mathematics curriculum.

Six language tests were given upon common language structures; they included the following:

1. "Large" Test: The first linguistic task presented the children with a card containing three numerals. The numeral in the middle was both physically larger and numerically largest and the child was asked to pick out the largest numeral. Then, a second card was presented. This time there were two "largest" numerals. The numeral in the center (3) was physically largest, and the

numeral on the left (5) was numerically the largest. The child was again asked to indicate which was the largest numeral.

The main purpose of this task was to find out if the child would choose the number that was physically largest, thus indicating that he/she interprets large in the most obvious, primitive way. If the child chose the number which was conceptually (numerically) largest, he was reflecting the fact that he/she was focusing on that aspect of "largest" which is mostly referred to when learning mathematics. He would be able then to disregard the more primitive designation of the term. The highest level of performance required that the child be aware of the fact that there are two ways of interpreting "largest," indicating that he can operate on the two levels simultaneously.

2. Conditional Sentence Test: In the second linguistic task, conditional sentences were presented. Two sentences included negative conditions, as:

   (1) Raise your hand if it isn't red.

   (2) Raise your hand if it isn't a circle.

   For the first sentence, a yellow block was indicated and the child was required to raise his hand. For the second sentence, a circle was shown and the child had to inhibit his reaction.

   Four positive conditional sentences were then presented. In the first two sentences, the condition was at the beginning of the sentence, as:

(3) If it is red, raise your hand.

(4) If it is square, raise your hand.

In the next two sentences, the condition was set at the end of the sentence, as:

(5) Raise your hand if it is yellow.

(6) Raise your hand if it is a circle.

For sentences 3 and 5, the condition was true and the child was to raise his hand. For sentences 4 and 6, the sentence was false and the child had to inhibit his action.

Four sentences containing "unless" were then presented. In the first two sentences, the term "unless" was placed at the beginning of the sentence, as:

(7) Unless it is an orange, raise your hand.

(8) Unless it is a woman, raise your hand.

In the next two sentences, the term being tested, "unless," was placed at the end of the sentence, as:

(9) Raise your hand, unless it is yellow.

(10) Raise your hand, unless it is a circle.

Again, the first sentences in these pairs (7 and 9) were true and the second of the sentences (8 and 10) were false.

The order of the sentences was varied in the testing situation to see if the location of the sentence in the series had any effect upon the answers. Since the child was expected to act in each situation, or perhaps, to fail to act, there was no doubt as to the interpretation of his response. Finally, the

words being used to base the action upon (circle, yellow, woman, etc.) were made clear to children before the testing situation began.

3. Following Sequential Order: The next eight sentences tapped the child's ability to follow directions given in varying forms of sequential order. The main question of interest was the child's ability to follow the directions correctly even when the order of activities in the sentences was different from the actual order in which the child was to carry out the instructions.

The construction of the first two sentences was simple and the order of actions in the sentences corresponded to the actual order of execution:

(1) Move a blue one, then move a red one.

(2) Move a square, then move a circle.

The next two sentences used the term "after" as the major sequence indicator. Again, the actual sequence of the activities was identical to the order of the actions in the sequence.

(3) After you pick up the cowboy, pick up the nurse.

(4) After you pick up the astronaut, pick up the doctor.

Cards with the given pictures were used; they were placed before the child in a convenient position for his/her use.

The sets of sentences included in the first and second tasks were considered to be easiest, as the order of the sequence in the sentence and the sequence of execution corresponded. The next two groups of sentences were assumed to be more difficult,

because of the lack of agreement between the sequence of the sentence and the sequence of the action.

The first variable in the next sentences was the term "before." Two sentences were used.

(5) Before you pick up the nurse, pick up the doctor.

(6) Before you pick up the astronaut, pick up the cowboy.

The next two sentences, which also reversed sentence and action order, included the term "after." They were:

(7) Move a ball after you move a square.

(8) Move a yellow one after you move a red one.

4. Passive Sentences: Twelve passive sentences were presented to the students in the next section of the language tasks. Four types of passive constructions were used. They were:

Sentences with an animate agent and an inanimate object:

(1) The woman was pushed by the cart.

(2) The boy was hit by the car.

(3) The boy was pushed by the chair.

Sentences with an inanimate agent and an animate object:

(4) The card was pushed by the woman.

(5) The car was hit by the boy.

(6) The chair was pushed by the boy.

Sentences with both agent and object animate:

(7) The boy was hit by the girl.

(8) The girl was hit by the boy.

(9) The cat was chased by the dog.

(10) The dog was chased by the cat.

Sentences with both agent and object inanimate:

(11) The car was hit by the bicycle.

(12) The bicycle was hit by the car.

The sentences were presented in random order, except that similar sentences were never presented sequentially. The main purpose of this section was to assess the child's ability to comprehend passive sentences with semantic cues to their meaning (sentences 1 through 6) and without semantic cues (sentences 7 through 12). In these last two sections, both the agent and the object can be acting or receiving the action.

To test the child's comprehension of these sentences, he/she was presented with two pictures for each sentence. The two pictures represented the two possible interpretations of the sentence and the child was asked to indicate which picture agreed with the sentence. For sentence 11, to give an example, the child was presented with pictures showing a car hitting a bicycle and a bicycle hitting a car. He had to choose which picture agreed with the sentence.

5. Relativization in Sentences: Four sentences were presented to test relativization in children. The following sentences were presented, each followed by the indicated question.

(1) The girls who played with Jim loved football. Who loved football?

(2) The dogs that chased the cat fell into a hole. Who fell into a hole?

(3) The girl Joe met liked the new teacher. Who liked the new teacher?

(4) The boys who met David laughed with the girls. Who laughed?

These sentences were considered difficult and late to appear in the child's ability to comprehend because the person doing the action (as the girls in sentence 1) and the activity (loved football in sentence 1) are separated by other agents that can be interpreted as the ones actually carrying out the activity. The order of the words and the proximity of the second agent to the activity make these sentences hard to comprehend for young children.

6. Mass and Count Nouns Knowledge: The last segment of the language usage task examined children's use of mass and count nouns. In the first part of the task, a story was developed with the child and during the telling of the story the child was asked questions of the form:

(1) How much bread did he eat?

(2) How many eggs did he eat?

(3) How much coffee did he drink?

(4) How many hot dogs did he eat?

(5) How much sugar did he put in the coffee?

(6) How many apples did he eat?

(7) How much lettuce did he eat?

(8) How many pieces of candy did he eat?

Or other examples utilizing the phrases "how much" versus "how many."

In the next part of the task the child was shown pictures of count nouns, such as apples, bananas, hot dogs, or doughnuts, as well as pictures of mass nouns, such as milk, cake, pie, soup, or ice cream. The child was then asked to phrase questions about the quantities in the pictures. A few trials were used for practice, so that the child understood the task.

This latter part of the task was considered the more difficult, as the child had to generate his own questions and could not rely on cues about the mass and count nouns, such as the one supplied by the tester in the earlier part when questions were asked of the child concerning the quantities.

H. Math Terminology Test: The second general form of language test was a mathematics terminology test given to first grade children. Bilingual children were tested over the terms in both English and Spanish, in order to ascertain their working knowledge of the terminology of math.

A Spanish-language mathematics terms list was examined and one hundred "most-used terms" were selected. The terms were those used most often in the primary grade mathematics curriculum. Children were then tested on their knowledge of these terms, in both English and Spanish.

I. Criterion-Referenced Arithmetic Test: In order to obtain a measure of the student's knowledge of number and numeration concepts and their proficiency in the basic operations, a criterion-referenced test was constructed over the objectives stressed for instruction in the primary grade mathematics curriculum. The items were taken from the pilot test forms of the Arithmetic

Basic Skills Test (ABST), a diagnostic mathematics test developed by the researchers. The items were selected on the basis of their relevance to the instructional objectives used in the curriculum over the two-year period of the study.

J. Precise Thinking Exercises:* Precise Thinking (PT) exercises consisted of a situation with a finite set of symbols (such as 3, a, 5, etc.), which constituted the alphabet of the situation, and finite rows of symbols taken from the alphabet.

Example 1: alphabet: 1,2,+
examples of rows: 12,
1+2
1+.

Example 2: alphabet: boy , the , ran
examples of rows: boy ran
the boy ran

Having a situation, patterns in the rows can be examined.

Example 3: Given: alphabet: 1,2,3;
Some of the patterns may be:
$P_1$: a row contains four symbols
$P_2$: a row starts with a 2.

A row may or may not conform to a given pattern. For example:

23 does not conform to $P_1$,
2313 conforms to both $P_1$, and $P_2$.

Given a situation and a pattern, two types of tasks emerged: the construction of rows which conformed to a given pattern (construction problems); and given a row, decide whether it conformed to a given pattern (decision problems). This was the basic concept underlying PT exercises. Following example 3, and given the pattern $P_3$: a row contains exactly

*For a detailed description of the materials, the reader is referred to a previous report #400-74-0074, June 1978.

3 symbols. Then the rows 123, 111, 233, 133, 321, 333, 3211, 2221, etc., can be classified by $P_3$ as rows which conform (123, 111, etc.) and rows which do not conform (3211, 2221). The situation was finite since the set of rows that conform was finite. Now the rows which follow the pattern can be arranged in the list 111, 112, ... 333. This list can now replace the pattern stated in verbal form. Lists are the best known forms of PT.

The activities selected for instruction were labeled "house & duck," "traffic light," "traffic flow," "cats & dogs" and "letter" exercises. They all dealt with instruction in following rules structure in algorithms which follow a precise thinking pattern.

The sequencing of the PT exercises was done using the following criteria:

1. Number of "rules" to be followed: exercises with a small number of rules were given before those with a greater number.
2. Number of manipulations required to complete the exercise: exercises which involved simple manipulations preceded those which required the student to perform a variety of operations.
3. Medium of the stimulus/response objects: exercises with pictorial representation were given to the students before activities which involved a more symbolic level.

Following these criteria, the PT exercises were administered to the students as follows:

1. "House/duck" exercise: pictorial representation; one manipulation per rule, and six rules.
2. "Simple letter" exercise: symbolic representation; one manipulation per rule, and eight rules.

3. "Traffic flow" exercise: pictorial representation; variability in the number of manipulations, and two rules. The idea of the reverse of an operation was introduced in this exercise.
4. "Cats and dogs" exercise: pictorial representation, at the most two manipulations per rule, and a more complex format for the stimulus materials (this exercise followed the concept of the Turing machine).
5. "Traffic light" exercise: pictorial representation of manipulative materials; symbolic (verbal) representations of the stimulus materials (rules), and an increase in the number of "rules" given. Also, the concepts of "start" and "good finish" were introduced.
6. "Complex letter" exercise: symbolic (verbal) representation of both manipulative objects and stimulus materials, and a significant increase in the number of rules and the manipulations required to complete each exercise (this activity follows the Turing machine concept).

## Case Study Data Analysis Procedures

Analysis procedures focused on the synthesization of the data obtained. As is customary in the reporting of case studies, an assessment of behavior was made and categorized; the existence (or lack) of patterns in behavior was investigated; and results were reported on the basis of how certain types of students responded or did not respond to certain variables. Tallies and per cents were used throughout the analysis. If appropriate, $X^2$ tests were run. The emphasis was placed on data analysis and not in hypothesis testing.

## Chapter Three

## CONCEPTUAL FRAMEWORK AND A REVIEW OF SELECTED RESEARCH WORKS

### Conceptual Framework

The primary purpose of the present study was to examine the manner in which monolingual (Spanish or English) and bilingual (Spanish/English) first and second grade students acquired concepts in the areas of number, numeration and the four basic operations. Assumptions were made at the inception of the study which provided direction in the development of a conceptual framework to be followed in the research procedures, the collection of data, and the reporting of the results of the research. Those assumptions were as follows (NIE, 1977):

1. Two interrelated strands of instruction in arithmetic appear from the earliest grades. These two strands, number and numeration, are different and distinct.
2. Since elementary mathematics is to a large extent a cumulative subject, it is apparent that children without adequate concepts of the decimal numeration system are at an increasing disadvantage as they encounter more sophisticated ideas in arithmetic.
3. There is some evidence that certain content areas in mathematics are more difficult for children from cultural, social and linguistic minorities.

In view of the above assumptions, the basic research approach used was similar to what Rosenberg (1976) calls "process evaluation, in which inquiry efforts are focused on actions rather than outcomes" and the

approach is directed toward "question raising research rather than conclusion drawing research."

The resulting conceptual framework involved the following elements:

1. Inquiry strategies: A case study approach with a focus on diagnosis was selected. The objectives for each case study were to describe the student's performance in the mathematics curriculum, review the content areas in which the child encountered difficulty in learning a given concept, and to describe in detail the pupil's performance in a series of Piagetian tasks.
2. Conceptual strands: The areas of number, numeration, and the four basic operations were selected as an area of inquiry, as they were taught in the primary grades, especially the first two grades.
3. Contextual variables: In order to define specific behaviors related to each of the conceptual strands, the objectives listed in the Dade County Mathematics Systems Program were utilized. These objectives represented the goals of instruction and were not dependent upon any specific textbook series.
4. Interacting variables: As behaviors were observed and noted, the need to explain and interpret the student actions arose. In order to facilitate the process, cognitive style, stage of cognitive development, and language dominance were selected as variables which might provide plausible hypotheses in terms of the students' acquisition (or lack) of mathematical concepts.

Figure 1 presents a diagram of the relationships between the components of the study's conceptual framework. The framework is not to be taken as a model of mathematics learning, but as the summary of the overall approach used in this study to investigate children's learning of mathematical concepts.

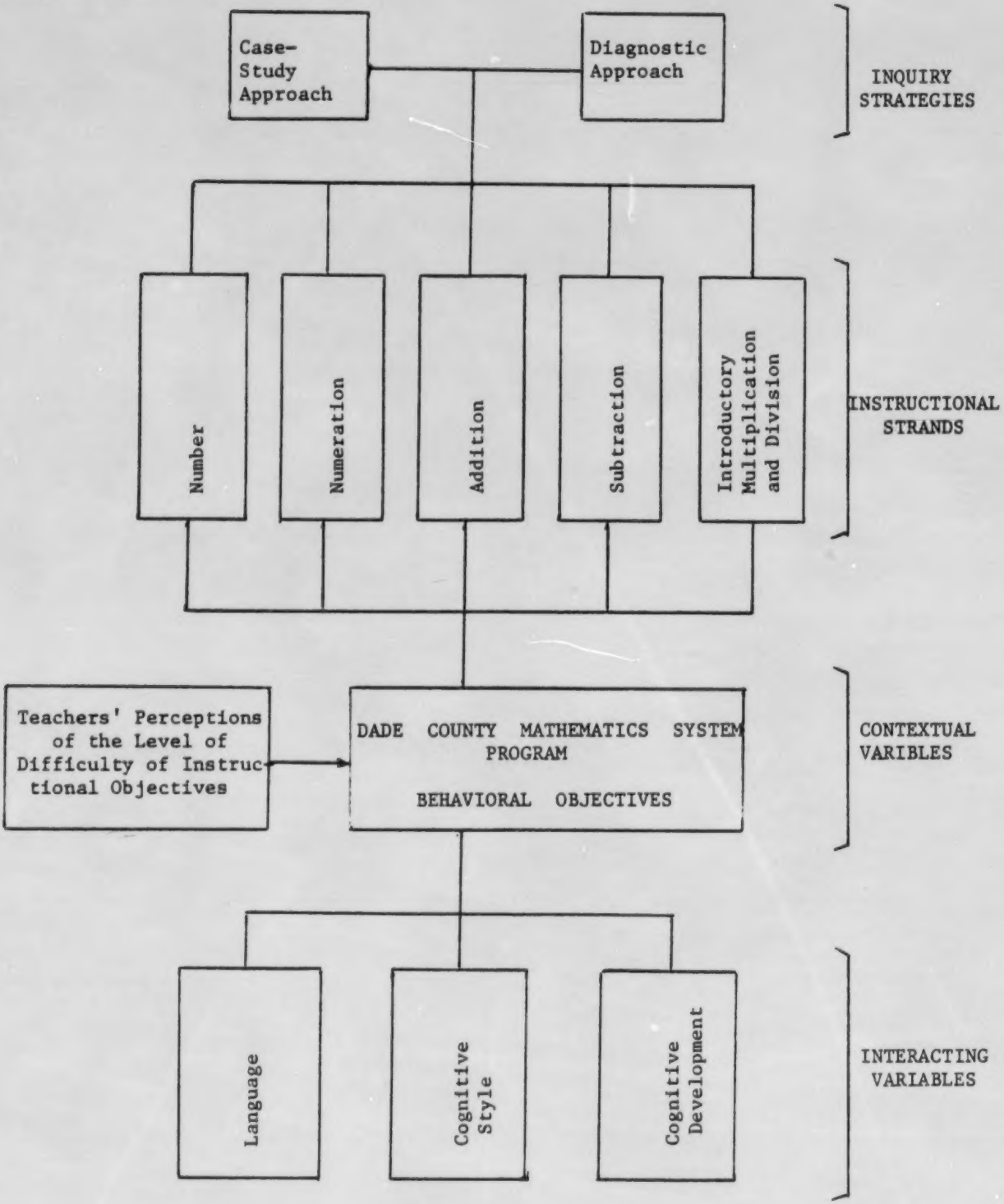


FIGURE 1

CONCEPTUAL FRAMEWORK FOR INVESTIGATION INTO MATHEMATICS LEARNING

## Review of Selected Research

Using the study's conceptual framework, the review of the literature is organized into sections, dealing with:

Cognitive style;

Language learning and its role in math concept acquisition;

Cognitive development and the learning of mathematics concepts in general.

Each of the sections may be divided into subsections.

### Cognitive Style

In a study conducted by Lysiak (1978) with middle school students of the Fort Worth Independent School District, results showed that Mexican-American and Black students were significantly lower in mathematics achievement than White Anglo students. It was hypothesized that a contributing factor to the low achievement was "... a culturally influenced cognitive style, that is incompatible with the typical methods used in teaching mathematics." The main reason for citing the study is to highlight two topics in which research is needed: the relation between cognitive style and mathematics achievement and cognitive style, ethnicity and teaching methods.

Witkin (1954) defined cognitive style as "characteristic modes of functioning that we show throughout our perceptual and intellectual activities in a highly consistent and persuasive way." Kagin, Moss and Siegel (1963) describe it as "stable individual preferences in the mode of perceptual organization and conceptual categorization of the external

environment." Both definitions seem to have obvious implications for the types of learning strategies available to students.

The cognitive style of field sensitive (dependent) or field independent students has been extensively researched. (Witkin, Dyk, Faterson, Goodenough and Karmp, 1962; Witkin, Lewis, Hertzman, Machover, Meissner and Wapner, 1954.) Field dependence has been operationally defined through the use of the Embedded Figures Test which requires the individual to detect simple geometric figures contained within more complex ones. The less able a person is to detect these figures the more field sensitive (dependent) he/she is.

In a large segment of studies conducted with college-age subjects, results indicated a relationship between field independence and high achievement in mathematics, science and engineering (Greenfield, 1971; Margolis, 1972; and Stein, 1968). In addition, other studies support the notion that field independent students show higher levels of school achievement (Bruinicks, 1968; Keogh and Donlon, 1972).

With respect to mathematics learning, Spitler (1971) states that:

> "... both mathematical ability and mathematical achievement are significantly related to field independence, furthermore, the ability to shift psychological sets in thinking and the ability to reason inductively on literal and numerical patterns is also significantly related to field independence."

Unfortunately, there is a lack of studies dealing with the relationship between cognitive style and mathematics achievement in the primary and middle elementary school grades.

## Cognitive Style and Ethnicity

Research by Ramirez and Castaneda (1974) indicates that "field sensitive (dependent) children do best on verbal tasks of intelligence tests, learn materials which have social content more easily, and they perform better when authority figures express confidence in their ability." On the other hand, "field independent children do best on analytic tasks, learn materials that are inanimate and impersonal, and their performance is not greatly affected by the opinions of others."

Ramirez's (1974) and other studies by Kagan and Madsen (1971) and Witkin and Berry (1975) indicate that socialization and cultural practices affect cognitive style. More specifically, Ramirez and Castaneda (1974) cite studies which indicate that Mexican-American students are more field sensitive (dependent) than Anglo students. The studies also indicate that these differences may be due to the Mexican-American children's desire to interact with others and belong to a social group, their display of sensitivity to others' feelings, and the willingness to rely upon adults for help and guidance. Ramirez and Price-Williams (1974) conclude that these factors may affect children's modes of organizing, classifying and assimilating information in school. This, in turn, has implications regarding a differentiation of teaching methods for the different cognitive styles.

## Learning and Cognitive Style

From the research of Witkin and his colleagues and Ramirez's work with Mexican-American students, the following are behaviors exhibited by field sensitive (dependent) and independent students as they go through the learning process. First, field sensitive students tend to:

- be motivated by forms of reward that offer personalized support, recognition or acceptance;
- learn materials which have a social or human content;
- be sensitive to the opinions of others and perform better when authority figures express confidence in their ability; and
- depend more on the external environment to define goals.

Field independent students, in contrast, tend to:

- do better than field sensitive students in analytical tasks;
- rely less on the opinion of others while performing tasks; and
- learn material which is inanimate and impersonal more easily than field sensitive students.

According to Ramirez and Castaneda (1974), the field sensitive behavior of many Mexican-American children can have a negative effect on their classroom performance since most commercial curricula appear to be overly biased toward the field independent style of learning. Cognitive style was utilized in the present study as a variable to facilitate the explanation of observed behaviors as primary grade children attempted to acquire mathematical concepts.

## Language and Learning

> "The acquisition of language does influence and enhance the child's thinking in a number of ways. When a child is able to link understanding of a situation with the use of right language, he/she has achieved high level control over that situation." (Jackson, et al, 1976.)

There is evidence that learning language and learning to use language have implications for the acquisition of concepts and the child's ability to think. The following section presents an overview of language acquisition theories, a review of the literature regarding language and thinking processes and a summary of studies concerning bilingualism and cognition.

## Language Acquisition

Two major theoretical approaches to language acquisition have been established since Skinner published his account on the topic in 1957. Skinner's extreme behaviorism triggered a reaction from N. Chomsky (1969), the linguist, who in turn stimulated the formation of what may be called a nativistic approach to language acquisition.

The behaviorist explanation of how language is acquired is basically straight-forward. It describes language behavior in the same terms behaviorism explains any other behavior. As such, learning a first language is governed by the same principles of reinforcement, punishment and imitation that govern learning in general. Children are exposed to verbal stimulation which serves as a model which they eventually attempt to imitate. Adults in the child's environment proceed to reinforce acceptable imitations and disregard or even punish inappropriate ones until the latter are weeded out and only correct language forms are used.

Under the assumptions of the behavioristic approach some hypotheses about language acquisition may be derived. For example, the order of pattern acquisition should reflect the frequency of patterns exhibited in the child's environment. Thus, children should learn the most frequent patterns of their language first and the less frequent patterns later.

No similarities are anticipated in the order of pattern acquisitions of different languages. In fact, differences should be anticipated even in the development of children learning the same language if there are marked differences in their linguistic environment. In addition, intentional manipulations of language development by the use of behavior management procedures should result in clearly predictable changes in language development.

In contrast, Nativists such as Miller, McNeil, Brown (1973) and others assume that humans are programmed to learn language as a unique human quality. Language acquisition therefore will not proceed as any other type of learning, but will follow certain rules of its own. Linguistic structures that are simpler and that require simple cognitive skills will be acquired early, regardless of their frequency in the language. More complex patterns that are cognitively more demanding will be acquired later. As a result, certain universal similarities are to be anticipated across languages and children around the world should display similar patterns in acquiring language. This prediction is also based upon the assumption that there are language universals which cut across languages even in adult speech. While the sequence of stages in development is expected to be fixed, clearly the rate of acquisition may differ from child to child. This sequence is fixed not only within the same language but also across languages.

The nativists also assume that language behavior is rule governed for children as well as adults, but that the rules for child language are different from adult rules. Nevertheless, they are rules. Rather than being passive absorbers of environmental influences, children are seen as active

processors of information. Children are constantly generating hypothetical rules about language use and examining the sample of language provided by adults in order to find support for their rules. Reinforcement and modeling will neither be able to explain and predict the language of young children, nor can it intentionally change the course of language development. Modeling and reinforcement are conceived of as sources of feedback for the young hypothesis testor, rather than processes exerting automatic control on language behavior.

Even though all nativists accept these general nativist ideas, they do differ in their conceptualization of what constitutes more or less complex patterns of language. Some, like C. Chomsky (1969) and Bloom (1970) base their judgement of complexity of N. Chomsky's generative transformation grammar of adult language, while Brown (1973) attempts an explanation based on Fillmore's case grammar. Traditionally the study of language is divided into three major fields: the sound system, syntax, and semantics. The two theoretical approaches are clearly represented in all three areas. Since the acquisition of the sound system is relatively irrelevant to this study of math, emphasis in this study will be on the development of syntax and semantics. It should be noted that the largest body of literature has been accumulated in the study of syntactic development. The data are increasing in the area of semantics as well, but their smaller scope reflects the natural difficulty of handling concepts of meaning.

Some Evidence on the Two Theories

The overwhelming evidence accumulated to date indicates that a behavioristic approach to language development alone cannot explain all the details

of language acquisition. There is no denying that imitation and reinforcement play a role in the process, especially in the acquisition of vocabulary. But these processes appear to be limited, especially where the acquisition of syntax is concerned. One must look to the nativist school to explain many of these details.

In looking at the role of reinforcement in the acquisition of syntax, Brown, Cazden and Bellugi (1964) observed that children are typically reinforced for the truthfulness of their utterances, not for their linguistic correctness. Also, studies testing the role of modeling and imitation indicate that correct syntactic modeling is not very powerful in changing the child's behavior. Children also produce many utterances which they have never heard from their adult models.

On the other side, there is clear evidence that children's language behavior is rule-governed. The over-generalization of rules is a well known phenomenon. Children apply newly acquired rules to exceptions which they used correctly before the new rule was acquired. For example, upon learning how to form the regular past tense, children produce such forms as "goed" or "comed," even though they used "went" and "came" correctly previously.

The acquisition of language seems to proceed in fairly well defined sequences, with children speaking different languages following a similar pattern. Early sentences are often called "telegraphic speech" because they contain only a few words and because these words are content words exclusively. Function words are omitted. Children of different languages show these patterns.

Psycholinguists that derive their explanations from a generative transformational grammar of adult language see child's early language as

reflecting mostly deep structure. As the child matures, transformations are gradually added to his language production. Hence, syntactic structures which require only a small number of transformations are correctly understood and used earlier than structures requiring a larger number of transformations. The stages that can be perceived in the use of negation and interrogation can serve to illustrate some tentative support to this approach. At an early stage, both questions and negation are formed by simply attaching negatives and question words to the beginning or the end of the sentence. Later these words are incorporated into the sentence but without the necessary auxiliary verbs which are added only at a later stage. Similarly, the passive is acquired in clearly defined stages reflecting the child's initial reliance on word order and his limited understanding of semantic concepts. In spite of the early belief that syntax is almost completely mastered at age five, C. Chomsky (1969) demonstrated that syntax continues to develop during the early school years.

In the area of semantics some evidence exists that even here there are well defined regularities in the acquisition of meaning that cannot be accounted for by the behavioristic model. One major example is the work of H. and E. Clark (1973) who demonstrated the existence of stage in the correct use of pairs of adjective antonyms with the unmarked adjective dominating the use and understanding of the pair for some time.

Other studies showed that count nouns are used correctly before mass nouns, that some semantic features are acquired earlier than others and that a change in the nature of word associations from mostly syntagmatic to mostly paradigmatic occurs in children between the ages of six and eight.

Language and Thought

The interest in the relationships between language and thinking is not a new one. One of the better known approaches was the Whorfien hypothesis which suggested that language determines and defines thought. Piaget, on the other hand, tends to see only a limited effect of language on thought. Some of his coworkers suggest that language development may be influenced by general cognitive development since some major changes in language mastery occur at times when major cognitive changes occur. Whatever the approach, language and thinking are probably strongly related. Thus the assessment of the child's level of language mastery may be important when we study his thinking in the area of math. Exploring the child's mastery of language may add to our understanding of his thinking and general cognitive ability at a certain time, hence, one of its uses in studying math learning in children. In addition it may be important to look at the language that teachers are using in the classroom to teach math. This may be important since clearly the child's language is not completely developed during the early years of elementary school and he may misinterpret the teacher's explanations and instruction if the structures he/she uses are not completely mastered by the child.

A relatively new area of interest in the relationships between language and thought is the school of sociolinguistics. Bernstein and Brandis (1970) suggest that the quality of the linguistic interaction between parents and children may be related to the child's intelligence and cognitive development. They suggest that an open interaction which encourages further mental exploration of issues by the child helps stimulate cognitive development to a further extent than a close-ended linguistic interaction.

Language and Math Learning

We would also like to examine the relationship between language and mathematics learning. It is our feeling that the language used to describe mathematical concepts is important when a child is learning math content, especially when a child is learning that content in a second language.

One of the quickest ways to illustrate the importance of language in math learning is to report the findings of a recent study by W. George Cathcart. Writing in the Journal for Research in Mathematics Education, Cathcart (1974) summarizes the results of a correlational study between math achievement and various non-mathematical measures. The single most important non-mathematical measure, according to Cathcart, was the child's listening ability. He states: "It could be that the instructional strategy was primarily a tell and do approach--one in which pupil attentiveness is an important variable." We might add one statement: It is the child's ability to listen and to understand which seems to be so important in predicting success.

Hargis (1977) reports considerable work with language in the various content areas; his work centers in the field of special education. He feels that while reading materials have been developed which match the child's language, math materials often present a mismatch between the vocabulary of the text and the child's vocabulary. It is Hargis' contention that all math ideas are embedded in definite language statements, with clauses signaling relationships and nouns signaling ideas.

Calhoun and Redmond (1974) make several key points about reading math materials. Math writing is characterized by its conciseness, its abstractness, and its complex relationships. In fact, there are "more ideas per

line" than in any other form of writing. Consider the following example from Calhoun and Redmond:

> When a mixed number is subtracted from a whole number, what number must you always borrow in the minuend?

Of the nineteen words in this sentence, 8 words (six words or phrases) have definite math meanings which must be understood to follow the meaning. They also point out that many math terms have alternate meanings, such as "set" being used conventionally as "set the table," or "order" being used conventionally as "this is an order." Children are often confused by such dual meanings, especially children who are working in a second language.

Consider the following figure presented by Calhoun and Redmond which indicates that every comparative statement, signaling either an equivalence or inequivalence relationship, has a definite type of phrase construction.

Figure 2

PHRASE CONSTRUCTION FOR COMPARATIVE STATEMENTS

| Concept | Non-equality | Equality |
|---|---|---|
| length | is longer/shorter than | is the same length as |
| width | is wider/narrower than | is the same width as |
| height | is taller/shorter than | is the same height as |
| area | has greater/less area than | has the same area as |
| mass | is heavier/lighter than | has the same mass as |
| time | takes more/less time than | takes the same time as |
| number | is greater than/is less than | is equal to |

The point is obvious: to master the concepts, the child must master the language of the concepts.

Another manner by which the importance of language can be stressed is to indicate that a recent review of the literature in just this area has been completed. Aiken (1971) writing in the Journal for Research in Mathematics Education, presents an excellent summary of the research dealing with language and number.

The emphasis we would like to place on language concerns the fact that language must be especially important for children whose dominant language is not English. The child who comes from a home where English is the main language spoken will have heard and used most of the comparative constructions which involve math concepts at some stage of his life. One cannot make the same assumption for the child whose native language is not English. This child may not be familiar with the "common" mathematics language.

## Bilingualism and Cognition

Another area which bears on the topic is the relationship between bilingualism and thinking. The major issue is: Does bilingualism help, hinder, or have no effect on cognitive ability? It is certainly not easy to answer this question. Most of the studies that have involved children who became bilingual at a very early age rely on data from individual children whose backgrounds and upbringings were very unusual. Probably the best large scale studies, conducted with children who became bilingual to some extent during the early school years, are those carried out by Lambert and his coworkers in Canada. The results of these studies suggest some

cognitive and affective advantages for early bilingualism, at least when both languages involved share the same level of prestige. It should be noted, however, that the nature and degree of bilingualism vary widely and it is not always easy to compare results of different studies in this area.

One contributing factor may be the great variety of definitions for bilinguality or bilingualism. The descriptions range from producing "... complete and meaningful utterances in the other languages..." (Haugen, 1953) to "...native control of the two languages." (Bloomfield, 1933.)

Studies with bilingual adults show that interrelationships exist between the two linguistic systems available to the individual and that to some extent, verbal messages are stored in memory at the level of meaning rather than in a specific language. In a study conducted by Feldman and Shen (1972) with bilingual Chicano and monolingual Black and Chicano students, the bilingual children outperformed the others in three Piaget-based tasks. The tasks consisted of recognizing an object after it had been physically altered (object constancy) and assigning labels to the objects in correctly constructed sentences. The authors argue that the ability of bilingual students to use arbitrary names for the objects is related to the fact that bilingual students see language as an instrument to be used according to a perceived linguistic and social context.

Peal and Lambert (1962) in a study dealing with intellectual development and school achievement, state that:

> "The picture that emerges of the French/English bilingual in Montreal is that of a youngster whose wider experiences in two cultures have given him advantages which a monolingual does not

> enjoy. Intellectually, his experience with two language systems seems to have left him with a mental flexibility, a superiority in concept information, and a more diversified set of mental abilities, in the sense that the patterns of abilities developed by bilinguals were more heterogeneous... In contrast, the monolingual appears to have a more unitary structure of intelligence which he must use for all types of intellectual tasks."

This type of evidence has not been recently contradicted (Lambert, 1976) and may stimulate further detailed studies into the nature of bilingualism and its cognitive correlates.

## Cognitive Development and Mathematics Learning

Since studies and theories in this area are vast, the discussion will be confined only to those topics related to the strands under study: number, numeration and the basic computational operations. Also, it is almost impossible to review the relevant literature without touching upon the work of Jean Piaget. Piaget's central argument in his theory of cognitive development deals with the quantitative and qualitative aspects of growth in thinking as children become older.

Jackson, et. al. (1976), in a summary of research (including Piaget's) related to children's development of thinking, state that:

1. Children's thinking matures without formal teaching. Even children who have never been to school eventually develop operational thinking. The rate of development may vary with the kinds of experiences and "push" offered by the child's environment.

2. Children learn best when they are ready to learn. Children whose minds have developed to the point of generating new ways of thinking make the transitions easily, while children who are not ready may learn only after prolonged effort.
3. The most important difference between children of high and low ability may be in the breadth and flexibility of their understanding at each stage, not in how fast they progress through the developmental sequence.

In relation to the acquisition of mathematical concepts, the research of Piaget offers a number of explanations. For the concepts of seriation and ordinal correspondence, Piaget states that children from five and one-half to seven years of age are able to solve problems involving serial correspondence using a perceptual method which involves trial and error. At around age seven, the child uses operation or intellectual procedures to solve the problem. At this age, the student considers the whole arrangement to be seriated rather than considering small groups within the collection. This has implications to the learning of the idea of ordinal number as it related to cardinal number: to determine how many objects are in a collection, the objects have to be ordered in some manner so they are counted only once. Piaget (1956) states that it is not until around 6 or 7 years of age that the child has a complete understanding of the relationship between ordination and cardination.

In relation to the acquisition of the concept of number, Piaget (1953) has stated that:

"It is a great mistake to suppose that a child acquires the notion of number and other mathematical concepts just from teaching. On

> the contrary to a remarkable degree, he develops them for himself, independently and spontaneously... Children must grasp the principle of conservation of quantity before they can develop the concept of number."

The concept of number has long been a concern for mathematicians and mathematical philosophers who investigate the nature of mathematics, for developmental psychologists who investigate the cognitive skills that presumably underlie numerical competencies, and for educators whose concern is constructing a curriculum for teaching numerical competence. Such a curriculum must be ordered in such a way as to parallel the development of cognitive skills in children.

Brainerd (1973) has reviewed major theories of natural number. The Peano-Von Neumann theory is an ordinal theory of number. Essentially the theory is that beginning with the primitive numeral, zero, you may then add "one" any number of times. As the individual continues adding "one" to zero each successive number thus produced is the successor of the primitive number and thus all the natural numbers can be produced by this series of operations. The consequence of this theory is that the child develops his concept of numbers sequentially, starting with the understanding of small numbers and gradually working upwards into larger numbers.

A second theory is the Russel-Frege theory which uses class as the primitive assumption in number. This theory assumes that a number has classes such that any two elements of a set are similar to each other and no elements outside the set are similar to any inside the set. Obviously, this brings us a set of classes representing each number in the number system.

A third theory is that of Piaget (also 1942, 1952). Piaget's theory operates with both the notion of order and of class. It is best presented in his work which introduces the concept of conservation of number. The classic ability of the concrete operational children's ability to conserve the physical aspect of quantity over a change, while the preoperational child cannot conserve the amount, is well recorded and discussed in education and psychology texts.

Each of the three theories has received criticism in terms of its mathematical validity. We are concerned here, however, with the psychological rather than the mathematical validity of these theories of natural number. Each of these theories predicts a series of sequentially developed concepts leading to a full understanding of the natural number system. These theories then predict a developmental order in which these basic numeral skills should be acquired by children.

In the field of education much attention has been directed towards the work of Piaget. Brainerd (1973), however, studied children in kindergarten and first grade and concluded that the preponderance of evidence suggests that the ordinal theory of number most closely predicts the order in which skills are actually acquired. The data, however, are by no means conclusive.

The practice, then, of listing the various developmental stages of Piaget and stating that these can guide the development of curriculum is reasoning along a rather simplistic line. In listing the various stages that Piaget identifies, the switch from the preoperational to the concrete operation stage at approximately the age of six is often cited as having

a major curricular impact. Preoperational children have problems with the learning of math concepts. But in the area of math the child is not always working exclusively at the preoperational or the concrete operational stage.

Flavell (1963), for example, reports several research studies done by Piaget and his associates that relate the difficulty in making assumptions about a child's stage of development. In working at the preoperational stage the child cannot conserve the quantity or amount across a physical change. Hence, when two sets of six items are put before the child, with one set spread apart to occupy a greater amount of space, the preoperational child will state that the two sets are not equal. One appears to be larger than the other. The concrete operational child will state the sets are equal. The problem is, however, that the conservation of amount does not generalize to all numbers immediately.

Flavell, citing studies done by Piaget and his associates, states that the conservation of amount is only gradually applied to larger and larger sets. The child may be in the concrete operational stage for numbers less than ten but in the preoperational stage for numbers larger than ten. Other research studies report findings which indicate that the child can understand selected math ideas when he cannot understand Piaget's conservation of number concept (Mpiangu and Gentile, 1975). Apparently the conception of number as defined by Piaget and other mathematical concepts represent various dimensions which can occur simultaneously in the child's thinking. Or stated succinctly, some math ideas apparently can be learned by children whose thinking is basically at the preoperational stage.

The complexity of development is further illustrated by Siegel (1971 a, 1971 b) who found that children's ability to recognize ordinal

position develops later than the equivalence attribute of conservation. This report of the equivalence attribute developing first is reversed by D'Mello and Willemsen (1969) who found that children were able to count the number of objects with another set having the same number of objects. We must conclude that the exact, empirical ordering of the acquisition of numerical skills is by no means well established.

Another experiment which indicates the difficulty of making generalized statements about the child's development of number concepts comes from the work of Piaget, as reported by Flavell (1963). In an experiment the child is given two sets, where set A is larger than set B, and asked to draw two equal subsets, one from each set. Yet because set A is larger than set B, the child infers that the subset of set A must also be larger than the subset of set B. Piaget concludes that evidently the child has problems in differentiating numerical attributes from logical attributes. Set A is logically larger than set B. Therefore, the subset of set A must be larger than the subset of set B.

It appears, then, that readiness studies must be made for mathematical concepts themselves. At what age can a child understand and work with the idea of number sentences? At what age can the child understand and work with the idea of the missing addend as part of a number sentence? Questions such as these are not immediately answerable from the existing literature on the development of number concepts in young children.

One additional idea helps to illustrate the problems children would have learning math concepts, especially when working in a second language. Ginsberg (1976), working from the case studies of several children and

from some of the older literature on counting, has made several general statements about the way in which children learn the sequence of counting.

He begins by suggesting that the numerals from 1 through 19 are basically memorized. This is a laborious, time consuming task that occupies many of the child's preschool years. Ginsberg states that the child will have many frustrating experiences attempting to ascertain what is meant by the adult who says "Count these!" or "Here are three cookies." Some time will elapse before the child is able to count items or list the numerals in order.

But once the child has learned the concept of counting through 19, Ginsberg hypothesizes that subsequent abilities are based upon the learning of specific rules. He feels that children do not memorize the order of numerals, instead, they obtain several general rules which can be used in working out the higher numerals. As evidence for this he cites that a child will count from twenty-one to twenty-nine and then will ask: "What comes next?" He will then count from thirty-one to thirty-nine and again will ask, "What comes next?" Ginsberg feels this demonstrates that the child is memorizing a general pattern or rule for constructing large numbers; he just needs to memorize the order for the numerals representing multiples of ten.

Ginsberg also uses two errors made by children in counting as evidence of the learning of basic rules. The child who counts "... twenty-eight, twenty-nine, twenty-ten" has learned the basic idea of counting, he just continues the use of a pattern too long. Likewise, the child who counts by ten and says "... eighty, ninety, tenty," has also learned a rule, but

again continues it for too long a period. Such thinking by a child, while wrong, is acceptable, according to Ginsberg. It shows that the child has identified the existing pattern. He simply needs to refine his thinking.

Learning the rules for counting would be a particular problem for the child whose predominant language is not English. While the native English speaker is exposed to examples of correct counting daily, the non-English speaking student will not hear daily examples of correct number formation. Yet for the child to follow standard math instruction, he must be competent in counting in English.

Development of the Concept of Numeration

Despite the lack of agreement as to the order in which the basic numerical skills emerge, there is little doubt that such skills must be acquired before an understanding of the decimal system can take place. Central to understanding the nature of the decimal number system is an understanding of the nature of symbolic substitution systems. Before algorithms for the operations performed on numbers may be learned the child must understand the relationship between a number, its numerical symbol, the place value system and those rules necessary to represent multiple place numbers.

The concept of place value involves symbolic substitution. That is, given the base of the number system each place represents n units of that place immediately to its right. Thus, in the tens place a two represents two times ten or twenty. A similar system may be built using analogs in which the number of the base varies. Research is available on such systems.

Research into the development of numeration systems takes several directions, but that which is important here is the study of numeration through the use of "other number base systems." Research into the use of other number base systems was popular in the early 1960's, immediately after the publication of the "new math." Most of the studies were inconclusive, with no significant difference between groups given instruction in base 10 and those given instruction in other bases, where knowledge of the base 10 system was the dependent variable. (See for example, Jackson, 1965; Lerch, 1963; McCormick, 1965 or Schlingsog, 1965.) The conclusion reached by most of these authors was that instruction in other number base systems "did not detract" from the learning of decimal systems.

Development of the Addition and Subtraction Algorithms

According to Landa (1974):

> "Algorithms have significance for instructional theory because, when we teach a student an algorithm for the solution of problems, we not only provide him with the means to control those objects which he will transform with the help of that algorithm, but also with the means to control himself, his own intellectual operations, and his own practical actions."

The teaching of algorithmic processes, or training in abstract algorithms, has been found to have a positive effect in the learning of the algorithms for whole number operations (Cuevas, 1975). With respect to the teaching of algorithms in mathematics, Landa (1974) is of the opinion that:

> "... the instructional process should not and cannot under any circumstances be reduced to teaching algorithms. Instruction

in algorithms is only one of the aspects of the teaching process, though a very important one."

Algorithms have three properties: specificity, generality and resultivity. The first property is concerned with the requirement that the directions in algorithms must be strictly and clearly defined. Generality resides in the fact that any member belonging to the defined problem may take the starting place in an algorithm. For example, the addition and subtraction of numbers is applicable not only to the numbers 327 and 24, or 57 and 3, but to any two natural numbers. Resultivity is reflected in the fact that an algorithm always converges on a specific sought-for result.

Rule-Governed Behavior

The learning of the concepts number, numeration and operations can be conceived as a generative process based on rule-governed behaviors, which can be viewed as types of algorithms. Rule-governed (R-G) behavior can be viewed as a function in the mathematical sense. There exists a one-to-one correspondence between stimuli and responses. It entails the ability to discern from a set of operationally distinct responses the appropriate one for each stimulus in a set of operationally distinct stimuli. Taking multiplication of whole numbers and using the given definition, the behavior is R-G if for every pair of numbers there exists a unique number called the product.

According to Scandura (1966, 1967, 1968, 1971), rules can be presented as an ordered triple, with components D, O, R. D is the set of stimuli which determines the responses; O is the operation or procedure which produces the responses for each corresponding stimulus (or stimuli);

and R is the class of functionally different responses. An operation or generative procedure consists of at most four kinds of rules: decoding rules, used to place stimulus properties into storage; transforming rules through which stored properties of stimuli are changed in storage; encoding rules which take these stored components and make them observable, and rules by which other rules are selected with a given goal and the outcome of a previously stored rule.

Church (1936) proposed that any mathematical behavior which can be produced by an algorithm is composed of these four types of rules. Scandura (1971) states that this would include most of the mathematical behaviors one would want to see in the elementary school age child; such as performing mathematical computations. Wittrock (1974) is of the same inclination. He states that:

> "... the learning of mathematics should be viewed as a generative process. This involves relating the structural organization to the learner's experience and encouraging the learner to process generatively the information."

Scandura's (1971) theory of R-G behavior in the learning of mathematics grew out of his feeling of a lack of existence of a theoretical framework in which mathematical knowledge and mathematical behavior may be viewed in a psychologically meaningful way.

Most of the mathematical behavior was being explained in terms of S-R mediation theory (Millenson, 1967; Suppes, 1969), but Scandura felt that:

> "... the network of S-R associations required to do the job would almost certainly be so complex as to provide little intuitive

guidance in formulating research on complex mathematical learning."

Scandura's theory for mathematics learning seems to be patterned after Chomsky's theories in linguistics. Chomsky (1957), introduced generative procedures and rules to explain (ideal) language behavior.

Other theorists (Ausubel, 1963; Dienes, 1963; Gagne, 1962; Piaget, as described in Flavell, 1963) feel that the S-R notation cannot convey the essence of meaningful learning. As a group, they express the opinion that an association or network of associations may be too restrictive to reflect all that a person does in a problem-solving situation.

It is Scandura's (1971) major thesis that:

"... rules are the basic building block of all mathematical knowledge and that, if looked at in the right way, all mathematical behavior is rule-governed."

## Summary

This chapter, reviewing related research and literature, has focused on the three main identified variables: cognitive style and the learning of mathematics; language acquisition and the learning of mathematics; and cognitive development and how it relates to mathematics learning. While it would be impossible, in one volume, to review all the research and literature in these areas, the researchers here have attempted to present a sample of the available material.

## Chapter Four

## ANALYSIS OF MATH CURRICULUM OBJECTIVES

### Content Analysis

Table V shows a summary of the analysis of instructional objectives covered in the Dade County Systems Mathematics Program. Objectives dealing with number, numeration and computational operations were listed for the content analysis in an effort to determine what is actually being taught under these headings. Concepts dealing with geometry, special numeration systems (money, time, etc.) and fractional numbers were not included in the analysis as they were beyond the scope of this study.

From the results presented in Table V, certain patterns in the structure of the systems curriculum become evident:

1. Number concepts are introduced at the early levels, and reviewed or expanded at later levels, through Level X.
2. Numeration concepts are distributed throughout the levels of the program, with emphasis at Level VIII, where expanded notation is presented.
3. Addition and subtraction concepts and algorithms are present in almost all levels. Emphasis on these operations is placed throughout the middle of the primary grade sequence, Levels V to VII for addition and VII to XIII for subtraction.
4. Multiplication and division concepts and algorithms are placed at the higher levels of the primary grade curriculum, Levels VIII and up. They are not finished in the primary grade levels. Division concepts are included in Level I, but are not stressed again until Level VI.

Table V

Summary of Analysis of the Dade County Systems Math Program: Number of Objectives by Level and Content Area

| LEVEL | CONTENT AREAS | | | | | |
|---|---|---|---|---|---|---|
| | Number | Numeration | Addition | Subtraction | Multiplication | Division |
| I | 9 | 2 | 2 | 1 | | 2 |
| II | 7 | 3 | | | | |
| III | 1 | 5 | 7 | 4 | | |
| IV | | 1 | 2 | 2 | | |
| V | 2 | 6 | 13 | 4 | | |
| VI | 2 | | 6 | 3 | | 4 |
| VII | 1 | 5 | 14 | 8 | | |
| VIII | 4 | 2 | 8 | 4 | 2 | |
| IX | 3 | 2 | 2 | 6 | 13 | |
| X | | | 3 | 2 | 2 | 13 |
| XI | | 3 | 5 | 6 | 10 | |
| XII | | 1 | 1 | 2 | 3 | 10 |
| XIII | | 8 | 1 | 6 | 10 | 7 |
| XIV | | | | | 1 | 2 |
| XV | | 2 | 1 | 4 | 2 | 5 |

A complete listing of all the objectives included in the content analysis is given in Appendix A. Objectives are listed by the following content areas:

1. Number Objectives
2. Numeration Objectives
3. Addition Objectives
4. Subtraction Objectives
5. Multiplication Objectives
6. Division Objectives

These objectives served as the curricular focus for the study, as the general model followed was to ascertain what is actually being taught under the headings of number and numeration, then survey teachers as to the difficulty level of the various items, and finally to observe and test children as they attempt to accomplish the objectives.

## Teacher Survey Results

An overall description of responses from items in the teacher survey was given in a previous report. Overall results can be seen there or are available from the researchers. It is the purpose of this discussion to move beyond a general description of the responses and to analyze those responses for patterns in two general areas.

The first area dealt with an analysis of objectives perceived as "difficult" or "easy" by a majority of respondents. Teachers were regarded as an untapped data source for descriptions of children's mathematical learning difficulties. The objectives perceived as being "difficult" by a

majority of teachers were given special attention by the researchers in their subsequent classroom visitations.

The second concern was to analyze those objectives receiving different ratings from teachers of English-dominant and Spanish-dominant children. The researchers wished to ascertain if there were differences in objectives perceived as difficult by teachers of the two language groups, and if language variables seemed to exist in the objectives rated differently.

All sets of objectives mentioned here were included in the ten month report and can be found in Appendix A. The tables include the objectives listed by content area, including the following areas:

1. Number Concept Objectives
2. Numeration Objectives
3. Special System Numeration Objectives
4. Addition Objectives
5. Subtraction Objectives
6. Multiplication Objectives
7. Division Objectives

One other technical point should be made. The response pattern utilized for the teacher survey was a five-point Likert scale, with a neutral response category included. While this increased the error of central tendency often found in Likert scale data, it also stressed that only those objectives standing out as being "difficult" or "easy" for children would appear on the objective lists. The objectives identified would be those which teachers perceived as most important.

Objectives Rated as Difficult By a Majority of Teachers

One particular question was asked about the objectives perceived as difficult by a majority of teachers. "Is it possible to categorize those objectives?" It would be important to identify areas which seemed to be of concern to a majority of teachers. If there were no general areas perceived as "difficult," then the difficulties must be randomly distributed throughout the entire systems program and were the result of no major causes.

Such a question was of major importance. If difficult objectives can be grouped, then major questions can be asked about those objectives. Why do they present difficulty? Are they misplaced in the curriculum when children's developmental patterns are considered? Are they rated as being "difficult" by teachers of differing age levels of children? The answers to such questions were of major importance to the concern of "how" and "when" math concepts should be taught.

An examination of the objectives did indicate that those objectives perceived and rated by a majority of teachers as difficult could be grouped into several specific areas. Let us consider each of those areas:

A. Number Line Objectives: The number line is often used as an explanation device for basic computational operations. Objectives concerned with the number line contain such items as "writing the addition number sentence shown on the number line" or "matching the number line expression to a subtraction number sentence." There were fifteen such objectives included in the primary grade levels of the Dade County Systems Math Program; thirteen of the fifteen number line objectives were perceived as difficult by a majority of teachers and rated accordingly. No number line objective

appeared on the list of objectives perceived as "easy" by teachers. The list of difficult number line objectives appears in Table VI.

Table VI

Number Line Objectives Perceived As Difficult
By a Majority of Teachers

| OPERATION LEVEL | OBJECTIVE | % OF TEACHERS RATING OBJECTIVE DIFFICULT |
|---|---|---|
| ADDITION: | | |
| III | Writes addition equation from number line, sums to 10. | 77.4% |
| | Translates number line representation into equation, sums to 10. | 82.3% |
| V | Writes number sentences from number line, sums 11-18. | 70.6% |
| VII | Matches examples of commutative property to number line. | 56.9% |
| | Writes equations: form a + b = b + a using number line. | 62.8% |
| | Completes addition equations, associate property, using number lines. | 80.4% |
| SUBTRACTION: | | |
| III | Matches number line to subtraction sentence, sums to 10. | 85.5% |
| VII | Subtraction as a non-commutative operation, using number lines. | 85.4% |
| MULTIPLICATION: | | |
| IX | Supplies number names of equal addends and related factors from number lines showing repeated addition. | 64.3% |
| XI | Addition and multiplication equations from number lines. | 58.4% |
| DIVISION: | | |
| VI | Partitions sets with number line drawing, totals equal to or less than 25. | 86.3% |
| X | Uses number line to report: N = __ sets of __. | 71.0% |
| XIII | Writes __ jumps of __ with __ remaining = N from number line for numbers 46 - 81. | 56.0% |

The objectives included in Table VI deal with expressions of all four basic operations on the number line. While some statements also include uses of the basic axioms of our number system, each objective in some way asks that the child represent something on a number line. The percentages of teachers rating these items as difficult should also be noted; they are the highest in the entire teacher survey. The percentage figures begin at well over 50% and extend just past 85%. It would appear that a large majority of teachers perceive the number line objectives as difficult for their children.

Two particular ideas should be mentioned. First, with a few exceptions, there is a tendency for the number line objectives to be rated as difficult by a fewer number of teachers as the level in which the objective is included increases. This would indicate that while number line objectives remain difficult for children into the second and third grade, they are not as difficult for as many children. This could be the result of learning how to deal with the number line, or it could indicate that as the child develops in his mathematical skill, the number line becomes easier to conceptualize and utilize.

Secondly, there is an amazing persistence of difficulty throughout the objectives which use the number line. While the rating percentages do generally decrease over the various levels, any new concept added to the objective immediately means that the percentage figure will rise. Note, for example, what happens to the percentage figures when the concepts of the associative and commutative properties are added to the number line objectives for addition and subtraction. The percentage of teachers rating

the items as difficult had been generally decreasing, but for these two new ideas the figures immediately jump back to the original levels.

Finally, the percentages of teachers rating these items as difficult were larger than those rating either the computational objectives or the operation definition objectives as difficult. One might conclude from this fact that the number line itself is more difficult to understand than the operations it is meant to explain. Does the number line include conceptual schemas which are difficult for children? One point which seems valid is that to understand the number line concept, the child must have the basic ability to conserve length over a physical change. He must recognize that greater distances on the number line represent greater amounts, and as one amount changes, the other must change also.

B. Missing Term Objectives: The second general area for which many items were identified as "difficult" for children dealt with the concept of missing terms in a number sentence. There were twenty-seven objectives in the primary levels dealing with missing terms; fifteen of those objectives were rated as "difficult" by a majority of teachers. The difficult objectives are listed in Table VII.

Table VII

"Missing Term" Objectives Perceived As Difficult
By a Majority of Teachers

| OPERATION LEVEL | OBJECTIVE | % OF TEACHERS RATING OBJECTIVE DIFFICULT |
|---|---|---|
| ADDITION: | | |
| III | Writes missing term (sum or addend), sums to ten. | 82.5% |
| V | Find missing terms, vertical form, up to 4 addends, sums to 10. | 68.6% |
| XIII | Missing terms, 2-3 digit addition, with regrouping. | 52.0% |
| SUBTRACTION: | | |
| III | Writes missing numeral, sum or addend, sums to 10. | 86.2% |
| V | Writes missing terms, horizontal and vertical forms, sums 11 to 18. | 78.0% |
| VII | Supplies missing terms, subtraction; 1 - 2 digits, no regrouping. | 59.2% |
| | Supplies missing terms, subtraction; 2 - 3 digits, horizontal and vertical form, no regrouping. | 52.1% |
| | Supplies missing term, subtraction; up to three digits horizontal form, no regrouping. | 69.4% |
| XIII | Missing terms, 2 - 3 digit, subtracting. | 74.0% |
| XV | Supplies missing term (4-6 digits) regrouping, horizontal notation. | 69.6% |
| MULTIPLICATION: | | |
| XV | Selects missing digit; multiples of 1 digit x whole number. | 60.8% |
| DIVISION: | | |
| X | Writes missing terms; form $a \div b$, products to 25. | 51.6% |
| | Writes missing terms, form $b\overline{)a}$, products to 25. | 58.1% |
| | Timed test, missing terms, $a \div b$ or $b\overline{)a}$, products to 25. | 54.9% |
| XIV | Timed test, supplies missing term, fifteen division exercises, 46 - 81, one minute. | 50.0% |

Table VII includes data on the topic of missing terms in an equation or problem for any of the basic operations. The objectives detail various difficulty levels of problems, as moving from one-digit problems to two- or three-digit problems. As can be seen in the table, these objectives have been rated as "difficult" by from half to 86% of the responding teachers.

Several different possibilities exist in the missing term problems. An addition equation of the form: 2 + 5 = __ is simply a basic fact; it would be difficult only in that the child must memorize a large number of such equations. If the child has mastered the concept of set union as a definition of addition, the task should be a memory task rather than a conceptual task.

Problems of the type: 3 + __ = 8, however, present an entirely different picture. In Piaget's terms, this problem constitutes a reversal. The child must change the problem from an addition to a subtraction problem in order to achieve an answer. As such, this is a new concept; there is ample literature stating that children of age six have difficulty with reversals.

For subtraction, three different forms are possible. The first form is similar to the first addition form; it presents the normal problem. If the child has mastered either the "take away" or the "comparison" definition for subtraction, then such a problem as: 7 - 4 = __ should not present conceptual difficulties. Again, it presents only the normal problems associated with memorizing the large number of "basic fact" items.

The missing subtrahend, however, presents greater difficulty. The form: 8 - __ = 5 calls for a reversal in order to solve the problem. As such, it presents the same conceptual problem as the missing addend presents in an addition equation.

The most difficult of all the missing term problems for subtraction, however, was the missing minuend problem, as: __ - 4 = 3. This process calls for a definite reversal of the operation; the child must add the two numerals given if he is to achieve an answer from the information given in the problem.

Obviously, both the missing minuend and subtrahend problems call for a conceptual skill which the basic subtraction problem does not call for. Again, the question can be asked: Does the child possess this additional reasoning capability at the age during which missing term problems are first presented? With large numbers of teachers indicating these objectives as difficult, some aspect of the problem is obviously causing difficulty.

Missing term objectives continued to cause difficulty in the operations of multiplication and division. In total, there were 27 objectives in the first fifteen levels of the systems program dealing with missing terms in equations or problems. Fifteen of these objectives were rated as "difficult" by responding teachers. Was there a pattern existing in these ratings?

One area of possibility concerns the location of the missing term; it could be located in two possible positions: (1) The missing term is placed in the "normal" position, as 6 + 4 = __. This would be a basic fact question. Four objectives, none of which were rated as difficult, covered problems presented in this manner. Therefore, of the twelve missing term objectives which were not rated as difficult by teachers, only eight utilized the second, more difficult form of the missing term problem.

The second form (2) placed the missing term as one of the two numerals which comprise the problem statement, as:

a. $6 + __ = 10$

b. $__ + 4 = 10$

The placement of the missing term in either of these positions, or similar positioning in problems dealing with other operations, made these problem types much more difficult. Yet, we still find that eight objectives dealing with missing term problems were not rated as "difficult" by responding teachers. Again, was there a pattern to the ratings?

All objectives occurring previous to or included in Level X were rated as difficult by responding teachers, with two exceptions. Both of these exceptions dealt with addition problems, where one term was missing from a series of three or more terms. Both objectives were included in Level VII, a level at which the child had had several previous experiences with missing addends. After Level X, however, virtually all objectives dealing with missing terms were not reported as difficult by the majority of responding teachers. There were four exceptions to this statement, each of which had an additional variable making the problem more difficult than similar objectives containing only missing terms. For example, in Level XIII, an objective using missing terms in two or three digit subtraction problems was not rated as difficult by a majority of responding teachers. Problems of the following type were used in this objective:

$$\begin{array}{r} 6\ 8\ 3 \\ -\ \square\ \square \\ \hline 2\ 5\ 1 \end{array}$$

A companion objective, however, was rated as difficult when it included problems using this form:

$$\begin{array}{r} 6\ \square\ 3 \\ -\ \square\ 3\ \square \\ \hline 2\ 5\ 1 \end{array}$$

Obviously, mixing the placement of the missing term made the problem much more difficult for the child.

We might conclude, then, that missing term objectives present problems only to younger children. This fact might suggest a developmental reason for the difficulty. Conceptually, the task might be beyond the conceptual schemas of first or second grade (six or seven year old) students. We see, from an examination of the suggested grade level placements for the various levels, that teachers perceive children through the first two grades as having difficulty with the objectives including missing terms, while children in the third grade or higher do not seem to have the same degree of difficulty. Only an objective containing a new or unique idea was rated as difficult after Level X.

C. Inverse Operation Objectives: The third general area of objectives perceived as "difficult" by a majority of respondents were those objectives dealing with "inverse operation." In all of these objectives, the child was asked to change one operation statement into the inverse operation statement, as, changing an addition number sentence into a subtraction number sentence. The objectives rated as difficult appear in Table VIII.

Table VIII

"Inverse Operations" Objectives Perceived
As Difficult By a Majority of Teachers

| OPERATION LEVEL | OBJECTIVE | % OF TEACHERS RATING OBJECTIVE DIFFICULT |
|---|---|---|
| ADDITION: | | |
| VIII | Selects addition equations from subtraction equations, sums equal to or less than 10. | 71.4% |
| | Selects addition equations from subtraction equations, sums 11 to 19. | 75.5% |
| X | Matches addition equations to subtraction equations, sums to 100. | 57.9% |
| SUBTRACTION: | | |
| VIII | Selects subtraction equations from addition equations, sums equal to or less than 10. | 74.5% |
| | Selects subtraction equations from addition equations, sums 11 to 18. | 76.5% |
| X | Matches subtraction equations to addition equations, sums to 100. | 60.9% |
| MULTIPLICATION: | | |
| X | Matches multiplication equations to division equations, products to 25. | 56.6% |
| DIVISION: | | |
| X | Matches division equations to multiplication equations, products to 25. | 66.9% |

Table VIII objectives deal with requiring that the child recognize addition and subtraction or multiplication and division as inverse operations. There are more than four such objectives because the objectives are separated by the magnitude of the answers. Note in the table that the results are surprisingly similar for each level; all Level VIII objectives received difficulty ratings from about 75% of the responding teachers, while all Level X objectives were rated as difficult by about 60% of the responding teachers. The consistency of the responses seems to indicate a strong feeling about the difficulty of the items by the majority of respondents.

Another point helps to show this consistency. There were nine inverse operation objectives in the systems program during the first fifteen levels. Eight of these were rated as "difficult" by responding teachers. Only one objective, found in Level XII and dealing with relating multiplication to division and vice versa, was not rated as "difficult" by over half the responding teachers.

The items presented in Tables VII and VIII are similar in nature. It can be hypothesized that the same basic conceptual skills are required for both items; the child must understand the nature of mathematical reversals to complete either group of objectives. Notice that there was a general trend of decreasing percentage figures for individual items as the level of inclusion increased. Such a response pattern would indicate that the items are perceived by teachers as generally becoming easier as the age of the student increases. This would support the contention that it might be a developmental problem which caused the objective to be rated as "difficult" by responding teachers.

D. Operation Definition Objectives: Another category of objectives rated as difficult were the various definitions for each basic operation. Table IX contains these objectives.

Table IX

Operation "Definition" Objectives Rated Difficult
By a Majority of Teachers

| OPERATION LEVEL | OBJECTIVE | % OF TEACHERS RATING OBJECTIVE DIFFICULT |
|---|---|---|
| ADDITION: | | |
| I | Identifies cardinal number of 2 sets and cardinal number of union of 2 sets. | 53.3% |
| SUBTRACTION: | | |
| I | Removes sets of specified cardinal number. | 60.0% |
| | Separate sets, identifies cardinal number of subsets. (Take away situation.) | 63.4% |
| | Identifies cardinal number of two sets, comparison. | 56.7% |
| MULTIPLICATION: | Cartesian Products: | |
| IX | Supplies multiplication sentences from arrays. | 51.7% |
| | Uses multiplication to name the number of possible pairings between elements of two sets, # of pairings equal to or less than 25. | 82.7% |
| | Organizes pairings of two sets to show multiplication situations, equal to or less than 25. | 79.3% |
| | Repeated Addition: | |
| IX | Interprets expressions, horizontal form, $a_1 + a_2 \ldots + a_n$ = n sets of a or n·a, products to or less than 25. | 51.7% |
| | Interprets expressions, vertical format, $a_1 + a_2 \ldots + a_n$ = n sets of a or n·a, products equal to or less than 25. | 55.2% |
| | Writes multiplication and addition number sentences, products and sums to 25. | 62.0% |
| | Supplies number names of equal addends and related factors from number lines showing repeated addition. | 64.3% |

Table IX (con't)

| OPERATION LEVEL | OBJECTIVE | % OF TEACHERS RATING OBJECTIVE DIFFICULT |
|---|---|---|
| DIVISION: | Measurement Division: | |
| I | Partitions set when given specified number in each set. | 63.3% |
| IV | Partitions objects into a specified number of equivalent subsets, n equal to or less than 12. | 75.0% |
| X | Matches repeated subtraction to division products, products to 25. | 77.4% |
| XII | Express partitioning of sets (26 - 45) as: N = __ subsets of __ with __ remaining. | 60.0% |
| | Partitions sets (26 - 45) as: N = __ subsets of __ with __ remaining. | 72.0% |
| | Matches repeated subtraction problems to division statements, 26 - 45. | 52.0% |
| | Partition Division: | |
| I | Partitions set when given specified number of subsets. | 70.0% |
| IV | Partitions objects into a specified number of equivalent subsets, n equal to or less than 12. | 76.8% |
| VI | Partitions set into N subsets of N with M elements, total equal to or less than 25. | 76.5% |

Table IX data present an interesting picture. The table shows the percentages of teachers rating an objective as difficult, where each objective listed is an example of a computational operation definition (set union for addition, repeated addition for multiplication, etc.) presented to a child at an age earlier than which he/she is expected to master that operation. In other words, the objectives are the spiral curriculum in operation. The child is presented with the concept of an operation at an early age, then is given simple, conceptual problems to work at a later age, and finally is expected to master the operation at a still later age.

As can be seen in the table, approximately half to three-fourths of the responding teachers rated these objectives as "difficult" for their students. This suggests that the objectives might be beyond the conceptual capabilities of the children at that particular age. Notice the levels at which the objectives are presented. Addition and subtraction definitions are presented in Level I, a kindergarten level. Multiplication is presented in Level V, a first grade level for the average achiever, and at Level IX, a second grade level. Finally, division is expected of the kindergarten child in a very simple manner, with more difficult objectives expected at the beginning of the second grade. In every instance the objectives were rated as "difficult" for children by their teachers.

Twenty-two objectives based upon operation definitions were included in the first fifteen levels of the systems program; twenty of these objectives were perceived as "difficult" for children by their teachers. The objectives include definitions for each operation; where multiple definitions exist, all are included in the objectives listed. When, for example, multiplication

can be represented by repeated addition or Cartesian products, both definitions were perceived as difficult by teachers rating objectives on these topics.

An interesting sidelight occurs for division. Whenever measurement and partition division were presented in the same level, partition division objectives were consistently rated difficult by a slightly greater percentage of teachers than were measurement division objectives. This is in line with existing research information.

The information here presents an interesting question: Do we present these objectives to children at such an age where they will automatically have difficulty conceptualizing the operation? This is a vital question. Presenting each operation too early would, for example, be a primary suspect in causes of math anxiety. Would children have a better view of their mathematics capabilities if the curriculum were delayed or modified slightly? Would fewer numbers of children have difficulty with math if the curriculum and presentation methods were modified? These questions will be addressed in the testing portion of this report.

E. Operation Translation Objectives: The next area of difficulty is a well documented area of mathematics for causing difficulty; we have described the general area as one of the skill of "translation." In essence, every objective identified in this area deals with translating a picture situation or a story situation into a mathematics number sentence. In other words, objectives in this area describe the types of objectives children have difficulty with in the general area of story problems. Objectives perceived as "difficult" by respondents in this area are listed in Table X.

Table X

Operation Translation Objectives Perceived as Difficult By a Majority of Teachers

| OPERATION LEVEL | OBJECTIVE | % OF TEACHERS RATING OBJECTIVE DIFFICULT |
|---|---|---|
| ADDITION: | | |
| III | Matches addition equation to picture union, sums to 10. | 62.9% |
| | Writes equations to addition situations, sums to 10. | 50.0% |
| SUBTRACTION: | | |
| III | Matches picture to related subtraction sentence, take away. | 70.5% |
| | Matches picture to related subtraction sentence, comparative. | 79.0% |
| VI | Matches subtraction word problem to equation, sums 11 to 18. | 63.5% |
| IX | Supplies or selects number sentences for pictured set subtraction scenes. | 55.1% |
| | Writes word problems for pictured subtraction situations. | 82.8% |
| | Determines number by which one set has more or fewer items and selects appropriate number sentence. | 75.8% |
| XI | Selects equation for subtraction word problems; 2 - 3 digits, no regrouping. | 66.7% |
| | Equations which are not translations of subtraction word problems, 2 - 3 digits, no regrouping. | 60.0% |
| MULTIPLICATION: | | |
| VIII | Translates word problems into equations and solve for products. | 54.6% |
| IX | Supplies expressions from pictured set situations, a x b, products equal to or less than 25. | 65.5% |
| DIVISION: | | |
| VI | Uses pictures to interpret division, total $\leq$ 25. | 63.1% |
| X | Writes equations and solutions from division word problems, products $\leq$ 25. | 64.5% |

For every objective in Table X the child is asked to look at a picture telling a number story, or read a number story, and translate that story or picture into a math sentence. Usually, he is asked to write or identify the correct number sentence for the picture or story. As can be seen by examining the table, fourteen such objectives were perceived by teachers as "difficult" for their students. The majority of these objectives dealt with subtraction.

In one possible interpretation, data presented in this table are not unique. Children's difficulties with story problems in math are well known and widely documented. However, there is one unique aspect of this particular section. In all other areas identified as difficult in this section of the report, the vast majority of objectives dealing with an identified topic were rated as "difficult" by responding teachers. That is not true in this case. There were thirty-four objectives dealing with translating stories and pictures into number sentences; only fourteen of those objectives were perceived and rated as difficult.

There was a pattern to those objectives which were perceived as difficult. An objective was usually perceived as difficult only if it is the first or near first objective that presents the translation task for a given arithmetic operation. In other words, the first addition objectives requiring children to deal with translation were likely to be rated as "causing difficulty"; subsequent objectives requiring this skill were not likely to be rated in such a manner by as many teachers. The obvious exception to this statement was the operation of subtraction. A great number of subtraction objectives were perceived as difficult and these ratings persisted through several levels of the systems program.

F. Numeration Objectives: The final general area containing several objectives rated as "difficult" by responding teachers was numeration. A basic point should be made about this topic, however. For most of the other areas identified, the majority of objectives in that area were perceived as difficult by teachers, except for the area termed "translation." For these topics, including translation, the objectives not identified as difficult did not appear on the list of objectives rated as "easy" for students.

The majority of numeration objectives, however, appear on the "easy" chart. Only a few objectives were rated as "difficult." Of the objectives so rated, one general topic from numeration stands out as containing most of them. It would appear that "expanded notation" is the one concept from the area of numeration skills which causes difficulty for children. These objectives appear in Table XI.

Table XI

Numeration Objectives
Perceived as "Difficult"
By a Majority of Teachers

| OPERATION LEVEL | OBJECTIVE | % OF TEACHERS RATING OBJECTIVE DIFFICULT |
|---|---|---|
| GROUPING: | | |
| VII | Gives cardinal number of sets grouped by 50's, 100's, for sets 200 to 1000. | 86.5% |
| EXPANDED NOTATION AND POSITION: | | |
| V | Writes expanded form for standard form. | 64.7% |
| VII | Supplies expanded numeral from pictures, 100 - 999. | 74.0% |
| | Interprets position value of specified digits, three digit numerals. | 68.7% |
| | Selects or writes standard form or other expanded form for three-digit numeral. | 50.0% |
| USING NUMERATION IN OPERATIONS: | | |
| XI | Selects best minuend, regrouping tens to ones, hundred to tens; 2 and 3 digit numerals. | 76.0% |

Again, let us repeat one point. Only a few numeration objectives were identified and rated as "difficult" by a majority of teachers; a great deal more were identified and rated as "easy" by a majority of teachers. The one exception to this statement is the concept of "expanded notation." Four objectives dealing with this concept were rated as difficult by the majority of respondents; the other two objectives in the table seem to be individual situations which cause difficulty.

The one area of concern to teachers, then, appears to be the use of expanded notation as a teaching device for place value concepts. It would appear that teachers feel children have some difficulty with this process.

As a comparison, those numeration objectives rated as "easy" by a majority of teachers appears in Table XII. Objectives in this table are also shown by general topics.

Table XII

Numeration Objectives Rated as "Easy"
By Teachers in the Teacher Survey

| OPERATION LEVEL | OBJECTIVE | % OF TEACHERS RATING OBJECTIVE EASY |
|---|---|---|
| READ AND WRITE STANDARD NUMERALS: | | |
| III | Writes standard numeral, 0 - 5. | 80.6% |
| | Writes standard numeral, 6 - 10. | 72.5% |
| V | Writes standard numeral from pictured collections, totals to 50. | 53.9% |
| IX | Writes standard numerals for written word names, 11 to 99. | 69.0% |
| XI | Matches numeral to written word name, 100 - 1000. | 68.0% |
| XIII | Supplies numeral for written number $\leq$ 1000. | 51.8% |
| RECOGNIZING LARGER AND SMALLER NUMERALS: | | |
| VIII | Identifies larger and smaller numeral, 11 to 100. | 74.5% |
| | Identifies smallest or greatest numeral in a triplet, 11 to 100. | 76.6% |
| XII | Identifies odd and even numerals, 1 to 100. | 62.5% |
| USING EXPANDED NOTATION: | | |
| XIII | Writes standard numeral or expanded form for (a x 10) + b or (a x 0) + b where a and b are less than ten. | 55.5% |
| MISCELLANEOUS: | | |
| I | Learner selects numeral cards to show number of things to ten. | 53.3% |
| VII | Counts by three to 18; five's to 20. | 52.0% |
| XIII | Rounds two-digit numerals to nearer ten. | 50.0% |

In conclusion, six general areas of difficulty were identified. They were:

1. Number Line Objectives
2. Missing Term Objectives
3. Inverse Operation Objectives
4. Operation Definition Objectives
5. Operation Translation Objectives
6. Numeration Objectives

The first four of these general areas were consistently rated as "causing difficulties" for students by teachers responding to the items in the survey questionnaire. The last two areas had a considerable number of "difficult" objectives, but many objectives in these general areas were not rated as "difficult" by teachers. Each objective listed in any area was identified by over fifty percent of the responding teachers as being difficult; a neutral category was included in the questionnaire which made it unnecessary for teachers to force their opinions into one view or another. General distribution patterns for those objectives identified as "difficult" usually had the remaining teachers selecting the neutral category as their response; very few teachers rated these objectives as "easy" for their students. Numeration was the one area which had exceptions to the general pattern.

Each of these areas, then, served as a focus for the researchers in their subsequent activities. Observations and interviews with children centered on objectives from these problem areas. Testing was done on these topics in order to ascertain reasons why they were rated as difficult. For each of the topics, certainly one possible hypothesis would be that the topic

is developmentally beyond the children when they are first required to master the listed objective.

Comparison of Results Between Teachers of English- and Spanish-Dominant Children

The second section of the analysis of teacher survey results compared ratings by teachers of English-dominant and Spanish-dominant children to ascertain if these teachers perceived the difficulty of required objectives differently. Results were separated into the two identified groupings, a Chi-square statistic was computed to test the significance of differences (.05 level) between the two groups, and a Cramer's V statistic was computed to test the strength of the relationship.

Unlike the first section of the teacher survey analysis, no real patterns were observed for obtaining significantly different ratings between the two groups of teachers. Thirty-six objectives were rated in different manners by the two groups; differences could be observed in the following directions.

1. 15 objectives were rated as "easier" by teachers of English-dominant children.
2. 4 objectives were rated as "harder" by teachers of Spanish-dominant children.
3. 1 objective was rated as "easier" by teachers of Spanish-dominant children.
4. 8 objectives were rated as "harder" by teachers of English-dominant children.
5. 8 objectives indicating miscellaneous differences in ratings between the two groups.

The grouping of objectives for these categories (1 through 4) are included in Appendix B; as there was only one discernible pattern in the groupings, they were not included in the body of this report.

The last category, miscellaneous differences, could be explained by several response patterns which did not indicate an important trend. One pattern was that teachers of Spanish-dominant children grouped themselves in the middle category, while teachers of English-dominant children chose both extreme categories. The reverse of this might also be true; teachers of Spanish-dominant children chose the extreme categories, while teachers of English-dominant children clustered in the neutral category.

Only one pattern emerged which needs to be discussed. Teachers of English-dominant children were more likely to rate a series of objectives dealing with number names as "easy" than were teachers of Spanish-dominant children. All of these objectives appear in Levels II and III of the systems program. One might conclude that for Spanish-dominant children, the task of learning the numerals in English, in addition to learning stated objectives of writing the numeral or matching it to its number equivalent, presents a slightly more difficult task for the bilingual child. Each objective actually includes two skills: the stated task and a language task. These objectives are listed in Table XIII.

Table XIII

Numeral Name Objectives Rated Differently
By Teachers of Different Linguistic Groups

| LEVEL | OBJECTIVE | CHI-SQUARE | CRAMER'S V |
|---|---|---|---|
| II | Matches set with cardinal number, 0 - 5. | 12.939 | .60803 |
| | Matches set with cardinal number, 6 - 10. | 13.052 | .61068 |
| III | Writes standard numeral, 0 - 5. | 15.980 | .54399 |
| | Writes standard numeral, 6 - 10. | 8.772 | .40306 |
| | Matches numeral to set, 11 - 20. | 9.833 | .43074 |

No other discernible pattern existed in the objectives identified as receiving significantly different responses from the teachers of the two language-dominant groups. One might conclude from the lack of patterns that most of the remaining differences were simply chance differences; when approximately four hundred comparisons are made at the .05 level of significance, approximately twenty will be significant purely by chance.

One might also conclude that teachers of the two language-dominant groups perceived math difficulties as lying in the area of math content, not in the area of language dominance. The one exception to this statement would lie in the early grades. For kindergarten and first grade students, there might be some skills for which the child has two needs: (1) he must learn the skill, and (2) he must learn the English language necessary for that skill.

## Chapter Five

## SUMMARY OF CASE STUDY DATA

Case study records were compiled for the majority of students in the sample; virtually complete case studies were obtained for 79 children, spread throughout the three schools involved in the study. Data were collected on variables such as cognitive development level (Piagetian), cognitive style, and overall comprehension of mathematics skills, such as number, numeration, and addition/subtraction concepts. In addition, all of the students were given speed and power tests in addition and subtraction. Standardized test scores (Metropolitan Readiness Test and Stanford Achievement Test) results were added to the case study records. To complete each case study, students were asked to perform selected Piagetian tasks to assess skills related to mathematics concept acquisition.

Table XIV presents a summary of the variables utilized in the analysis of the case study data. The table also indicates the sequence in which the variables are examined in this chapter of the report.

Analysis procedures focused on synthesization of the data gathered. As is customary in the reporting of case studies, assessment of behavior were made and categorized, the existence (or lack) of patterns in behavior was investigated, and results were reported on the basis of how certain types of students responded or did not respond to certain variables.

Table XIV

Summary of Variables Contrasted

In The Data Analysis

| CONTRASTING VARIABLES | CONTRASTED VARIABLES: COGNITIVE DEVELOPMENT LEVEL | QUANTITATIVE TEST (SAT) PERFORMANCE | COGNITIVE STYLE RESULTS | NUMBER-NUMERATION CONCEPTS | ADDITION/SUBTRACTION CONCEPTS | ADDITION/SUBTRACTION COMPUTATION |
|---|---|---|---|---|---|---|
| A. COGNITIVE DEVELOPMENT LEVEL | | X | X | X | X | X |
| B. COGNITIVE STYLE GROUP | X | X | | X | X | X |
| C. RELATIVE LANGUAGE PROFICIENCY | X | X | X | X | X | X |
| D. GRADE/AGE LEVELS | X | X | X | X | X | X |
| E. METROPOLITAN READINESS TEST PERFORMANCE | X | X | X | X | X | X |

NOTE: An "X" indicates the comparisons used in the analysis of the data.

Cognitive Development (Piagetian)

The students were first assessed as performing at a preoperational, transitional, or concrete operational stage. The data were examined on the basis of each developmental group's performance, and the results were as follows:

1. A larger percentage of students classified as transitional (54%) and concrete operational (56%) were identified as field independent than were those students classified as preoperational (31%).
2. Students at the transitional and concrete operational stages tended to score more frequently above the 7th stanine for the Stanford Achievement Test (SAT) mathematics comprehension subtest. 75% of the students classified as transitional scored above the 7th stanine, as did 56% of those classified as concrete operational. On the other hand, only 42% of those classified at the preoperational stage scored above the 7th stanine for the comprehension subtest of the SAT.
3. A larger percentage of the students at the preoperational stage (31%) were found with scores below the 5th stanine than were students at the transitional (8%) of concrete operational (5%) stages, again for the SAT subtest dealing with mathematics comprehension.
4. For the SAT computation subtest, 58% of the students at the transitional stage and 50% of the students at the concrete operational stage were found to have scores above the 7th stanine, while only 36% of the students assessed as preoperational scored at that level.

5. One hundred percent of the students at the transitional and concrete operations stages were found to have mastered number concepts skills. (Mastery was defined as answering 75% of the questions correctly in that subsection of the test.) On the other hand, only 81% of the students at the preoperational level displayed mastery of the number concepts.
6. In the area of numeration concepts, 83% of the students at the concrete operations stage and 74% of the pupils at the transitional stage showed mastery. (Again, mastery was defined as answering 75% of the questions correctly.) Only 63% of the preoperational students were found to have mastered numeration concepts.
7. A similar result was obtained for mastery of addition/subtraction concepts. 61% and 56% of the students at the transitional and concrete operational stages, respectively, were found to have mastered the concepts. A smaller percent (31%), of pupils at preoperational stage showed mastery level performance.
8. Table XV contains the correct items (mean scores) obtained by students in computation speed and power subtests for addition and subtraction.

Table XV

Summary of Computation Mean Scores
For Cognitive Development Levels

| OPERATION | COMPUTATION TYPE | PREOPERATIONAL | TRANSITIONAL | CONCRETE OPERATIONAL |
|---|---|---|---|---|
| Addition | Speed | $\bar{X}$ = 21(70%)* | $\bar{X}$ = 19(63%) | $\bar{X}$ = 23(77%) |
| | Power | $\bar{X}$ = 9(75%) | $\bar{X}$ = 9(75%) | $\bar{X}$ = 9(75%) |
| Subtraction | Speed | $\bar{X}$ = 16(53%) | $\bar{X}$ = 14(47%) | $\bar{X}$ = 17(57%) |
| | Power | $\bar{X}$ = 6(50%) | $\bar{X}$ = 6(50%) | $\bar{X}$ = 7(58%) |

*The percent of correct responses represented by the mean score is given in the parentheses.

Emerging Patterns

From the results shown above in items 2 through 7, there seems to be a consistently higher performance level from those students at the transitional or concrete operational level than for the students at the preoperational level. This higher level of performance was true for such topics as comprehension and computation for the SAT, number and numeration concepts from the ABST, and addition/subtraction concepts from the ABST. Two points should be made. First, higher percentages of students at the transitional and concrete operational stages displayed satisfactory performance than those students at the preoperational level. Secondly, while fewer preoperational students displayed mastery of the tested skills, it should be noted that sizeable number of preoperational students did perform adequately on the test sections. This would indicate that the Piagetian stages were

not absolute indicators of mathematics achievement. However, Piagetian stages did indicate trends in the achievement levels of students.

As shown in item #7, there was a sizeable drop in the number of students showing mastery of addition/subtraction concepts. Item #8 indicated a sizeable drop in the ability of students to perform subtraction operations as opposed to addition. An analysis of the individual items in the addition/subtraction concepts portion of the ABST (where general finding #7 was obtained) indicated that children were more likely to obtain correct answers for the addition questions than they were for the subtraction questions. This would indicate that children in the primary grades, especially the first grade, had more difficulties with subtraction than addition.

As can be seen in Table XV, students at the various developmental levels tended to perform about equally well in addition and subtraction power tests. All students seemed able to perform computational operations equally well. It would appear that the Piagetian stages do not differentiate between students in their ability to perform computational operations as well as they differentiate between students' conceptual ability.

## Cognitive Style

The Children's Embedded Figures Test (Witkin, et. al.) was used to assess students as field sensitive (dependent) or as field independent. Field sensitive (FS) students were identified as those scoring seven or less on the Embedded Figures Test; field independent (FI) students were

identified as those scoring thirteen or higher on the Embedded Figures Test. The data was analyzed on the basis of two cognitive style groups (FI and FS), and the results were as follows:

1. More (40%) of the field sensitive students scored at the 7th stanine or above in comprehension of mathematics concepts than in mathematics computation (25%). (Both tests were subtests of the SAT.)
2. More field independent students scored at the 7th stanine or higher in mathematics computation (77%) than in mathematics comprehension (54%).
3. 61% of the field sensitive students were found to be at a preoperational developmental stage, while 33% were assessed as transitional and only 5% as concrete operational.
4. 23% of the field independent students were operating at the preoperational level, as opposed to 46% at the transitional level and 31% at the concrete operational level.
5. FS and FI students performed similarly in the number and numeration concepts portions of the ABST. 85% and 75% of the FS students demonstrated mastery of the number and numeration subtests respectively, while 93% and 64% of the FI students demonstrated mastery on the respective tests. (Mastery was defined as a score of 75% or better.)
6. The largest discrepancy in performance between FS and FI students was found on the addition/subtraction concepts test. Only 30% of the FS students showed mastery of the concepts, while 64% of the FI students obtained the required score.

7. Table XVI presents the number of correct items (mean scores) obtained by FS and FI students in the computation speed subtests for addition and subtraction. As can be seen, for the addition power test, both groups had similar mean scores. For all other tests, the FI students performed slightly higher than the FS students.

Table XVI

Summary of Computation Mean Scores
For Cognitive Style Groups

| OPERATION | COMPUTATION TYPE | COGNITIVE STYLE GROUP FIELD SENSITIVE | FIELD INDEPENDENT |
|---|---|---|---|
| Addition | Speed | $\bar{X}$ = 21(70%)* | $\bar{X}$ = 23(77%) |
| | Power | $\bar{X}$ = 9(75%) | $\bar{X}$ = 9(75%) |
| Subtraction | Speed | $\bar{X}$ = 17(57%) | $\bar{X}$ = 20(67%) |
| | Power | $\bar{X}$ = 6(50%) | $\bar{X}$ = 8(67%) |

*The percent of correct responses represented by the mean score is given in the parentheses.

Emerging Patterns

Results shown in items #1 and #2 indicate that a larger percentage of FI students were performing at the 7th stanine or higher in both math comprehension and computation. The largest difference in scores was between the 77% of the FI students who achieved a 7th stanine score in computation and 25% of the FS students. For comprehension, the difference

was 54% to 40%, a much smaller difference between the two groups in the number of students performing at the 7th stanine or higher.

It is also interesting to note in items #1 and #2 that the direction of the high percentage figures was different. The larger number of FS students did well in the comprehension subtest, while the larger number of FI students did well in the computation subtest. This was in agreement with the theoretical literature, which implied that FI children should do better on tasks such as computational algorithms which demand rule-governed behavior.

From items #3 and #4 it can be seen that the FI students tended to be labeled consistently higher in their performance on the Piagetian tasks. Consequently, FI students were placed at higher developmental stages. The majority of FS students were rated as preoperational, while the largest percentage of FI students were rated as transitional.

From items #5, #6, and #7 it can be seen that while FI students sometimes performed at higher levels than FS students in number, numeration or addition/subtraction concepts, the performance was not higher for all subtests. On the other hand, FS students outperformed the FI students on one subtest only. With but one exception, FI students tended to outperform FS students on addition and subtraction concepts and computation, but both groups performed about the same on tests of number and numeration concepts.

Precise Thinking Exercises

In order to examine the influence of cognitive style on strategies students used while following algorithms, the Precise Thinking (PT)

exercises were devised. The objectives for this component of the overall research study were as follows:

1. To determine if PT exercises, as originally designed for fifth-grade students, were appropriate for use with first-grade students.
2. To investigate if there were any differences in performance on PT exercises between field sensitive (FS) and field independent (FI) first-grade students.
3. To determine the nature of the differences (if any) in performance on PT exercises between FI and FS students.

The overall purpose of the study was to gather data for each of the above questions so further research questions might be generated for study. No attempt was made to provide conclusive answers, nor to test hypotheses.

In the "House/Duck" exercises, any initial difficulties the students had in rule identification and manipulation of cards were remedied in subsequent exercises of the activity. FS students seemed to require more verbal reinforcement from the experimenter in each cycle of the exercises. This was evident by the student's pointing to the rules and waiting for the experimenter to give approval, or asking for a comment at the completion of an exercise. Most FI students required a minimum of assistance in completing the activity.

For the second activity ("Simple Letter"), it was readily apparent that the skills the students acquired in the "House/Duck" exercises transferred quickly to the second set of tasks. Most of the students knew what procedures to follow without having to be told by the experimenter. Also, both FS and FI students were able to guess the outcome of the final exercise in the activity.

The "Traffic Flow" exercises presented no major problems for either FS or FI students. All were able to complete the activity without difficulty. It was apparent from observations that FS students felt the need to trace the appropriate route from one given city to the next before selecting the correct sequence of "travel" cards. FI students were able to select the cards after visual examination of the route.

The "Cats and Dogs" exercises introduced the students to the concept of the Turing machine. (This was not mentioned to the children.) Most FS students had difficulty differentiating the "given" and the "result" of a selected rule from the overall set of rules. Also, many had problems in selecting the appropriate rule. Verbal reinforcement was requested by FS pupils throughout the activity. The only major problem which was apparent in the performance of the FI students was an observed difficulty with multiple card manipulations as elicited by selected rules. Other than that, most FI students did not appear to have difficulties completing the exercises.

The verbal/written format of the rules for the "Traffic Light" exercises presented some problems to FS students. Overall, there was an observed difference between FS and FI students in the areas of rule identification and card manipulation, with FS students having most of the problems. In this activity a required brown color card was purposely not provided; in its place an orange card was given. All students were able to handle this problem with equal ease. After finding no brown card, they selected the given orange card with little or no guidance from the experimenter.

In the last activity of the PT exercises ("Complex Letter"), the most marked differences in performance were observed. Students from School A (both FS and FI) became frustrated with the complexity of the rules, the medium (verbal/written) of the instructions, and the multiple card manipulations required in each cycle of the exercises. All of the pupils failed to complete the exercise. This was also true for FS students from School B. In this school, FI students had minor problems with "given/result" differentiation. These problems were remedied in subsequent exercises in the activity. FI students at School B were the only students able to complete all of the tasks given.

## Relative Language Proficiency

The Hispanic students from participating schools were given the Language Assessment Scales (De Avila and Duncan, 1975) in order to assess language skills. The students were classified as Spanish monolingual, bilingual, or English monolingual according to the criterion shown in Table XVII.

Table XVII

Scoring Scheme Used In The Language Classification

| LANGUAGE ABILITY* | LANGUAGE PROFICIENCY SCORES LEADING TO STATED CLASSIFICATION |
|---|---|
| Bilingual | 5/5, 5/4, 4/5, 4/4, 5/3, 4/3, 3/4, 3/5 |
| Monolingual (English or Spanish) | 5/1, 5/2, 1/5, 2/5, 4/1, 4/2, 1/4, 2/4, 1/3, 2/3, 3/1, and 3/2 |

*As measured by the Language Assessment Scales

The data were examined on the basis of the three language groups, with the following results:

1. A large percent of English monolingual students (62%) were found to be field independent, while 36% of the bilingual and 25% of the Spanish monolingual students were found to be field independent.
2. There was a larger percentage of field sensitive Spanish monolingual (75%) and bilingual (63%) than field independent students for both groups.
3. No findings in mathematics comprehension or computation from the SAT can be reported for Spanish monolingual students, as their English proficiency was judged "too low to take the SAT." The judgement was made for the school district by the classroom teacher.
4. No apparent difference in performance between bilingual and English monolingual students was found in the mathematics comprehension and computation tests, as measured by subtests of the SAT.
5. No apparent difference in performance between the comprehension and computation subtests was found for bilingual students.
6. No difference was found between Spanish monolingual, bilingual, and English monolingual groups in the distribution of students by stages of Piagetian development. A $X^2$ was computed; it was found to be nonsignificant at the .05 level.
7. A larger percentage of bilingual students showed mastery of number (100%), numeration (80%) and addition/subtraction concepts (40%) than Spanish and English monolingual students.

8. Spanish monolingual students showed the smallest number of students mastering number (75%), numeration (38%) and addition/subtraction concepts (13%) of all three language groups.
9. Table XVIII shows the mean number of items correct in the addition and subtraction computation tests for each of the three language groups. As can be seen, bilingual students seemed to score higher in both speed and power tests. Spanish- and English-monolingual students scored approximately the same for all tests.

Table XVIII

Summary of Computation Mean Scores
For Language Groups

| OPERATION | COMPUTATION TYPE | LANGUAGE GROUP SPANISH MONOLINGUAL | BILINGUAL | ENGLISH MONOLINGUAL |
|---|---|---|---|---|
| Addition | Speed | $\bar{X}$ = 13(43%)* | $\bar{X}$ = 23(77%) | $\bar{X}$ = 14(47%) |
| | Power | $\bar{X}$ = 6(50%) | $\bar{X}$ = 10(83%) | $\bar{X}$ = 8(67%) |
| Subtraction | Speed | $\bar{X}$ = 8(27%) | $\bar{X}$ = 21(70%) | $\bar{X}$ = 11(37%) |
| | Power | $\bar{X}$ = 5(42%) | $\bar{X}$ = 8(67%) | $\bar{X}$ = 6(50%) |

*The percent of correct responses represented by the mean is given in the parentheses.

Emerging Patterns

As can be seen in items #1 and #2, Spanish-monolingual students and bilingual students tended to be predominantly field sensitive, while Hispanics who were rated as English-monolingual in their language proficiency were found to be predominantly field independent. It is also interesting to

note that those students rated as Spanish-monolingual were more likely to be field sensitive that were students rated as bilingual. This would be an interesting addition to the research of Ramirez and Castaneda, who found that Hispanic students tended to be field sensitive. One hypothesis might be that the more contact the child has with English (both the language and the culture), the less likely it is that he/she would be field sensitive.

While bilingual Hispanics and English-monolingual Hispanics tended to perform similarly on the SAT's math sections, for all other tests the bilingual student outperformed both the English- and Spanish-monolingual student. As can be seen in Table XVIII, on the speed tests this difference was quite pronounced. The lowest performance was shown by the Spanish-monolingual student. Performance by all three groups on the addition/subtraction concepts test, however, was lower than for other groups.

## Grade/Age Level Performance

In order to analyze the students' performance on the mathematical concepts identified for the study on a developmental basis, the data were grouped by the child's grade and age levels within each grade. The children were divided according to the levels stated in Table XIX.

Table XIX

Grade and Age Level Divisions

| GRADE | GROUP | MINIMUM AGE | MAXIMUM AGE |
|---|---|---|---|
| First Grade | Group A (young) | 6 years, 6 months* | 6 years, 9 months |
| | Group B (old) | 7 years, 3 months | 7 years, 6 months |
| Second Grade | Group A (young) | 7 years, 6 months | 7 years, 9 months |
| | Group B (old) | 8 years, 3 months | 8 years, 6 months |

*All ages computed from the student's birthdate to June, 1979, the end of project date.

Information was collected and examined using only the identified groups. It should be noted that the age ranges excluded the children of average age; only the oldest and youngest students in a grade were included. The following general results were obtained.

1. 54% of the first grade students in group B (older) were assessed as field independent, in contrast with 18% of the students in group A (the younger group) in the first grade.
2. 62% of the second grade students in age group B (older) were found to be field independent, as opposed to 45% of the group A (younger).
3. In the first grade, 70% of the "younger" students scored above the 7th stanine in the SAT mathematics comprehension and computation subtests. A smaller percentage (60% of the comprehension test, 50% for the computation test) of the "older" first grade students scored above the 7th stanine in the math sections of the SAT.

4. The opposite trend was evident in the second grade. 56% of the older students scored above the 7th stanine in both comprehension and computation, while only 17% of the younger students obtained this score in the comprehension subtest and 33% in the computation subtest.
5. In terms of cognitive development, 9% of the "young" first graders were classified as being at the concrete operations stage, while 31% of the older students were found to be at this stage. For the second grade group a similar trend was found. 75% of the older students were assessed at the concrete operations stage, while only 25% of the younger students were at this stage. It should be noted that all students were tested during their first grade year.
6. Table XX presents the results of the students' mastery of number, numeration and addition/subtraction concepts. As can be seen in the table, the performance of both groups was about the same for number concepts; older students tended to outperform younger students on the numeration test; and younger first grade students tended to outperform older students on the addition/subtraction concepts test; while older second grade students outperformed the younger group on this test.

Table XX

Summary of Mastery Scores For Number, Numeration and Addition/Subtraction Concepts for Grade/Age Groups

| SKILL AREA | GRADE ONE | | | | GRADE TWO | | | |
|---|---|---|---|---|---|---|---|---|
| | YOUNG | | OLD | | YOUNG | | OLD | |
| | PASS | FAIL | PASS | FAIL | PASS | FAIL | PASS | FAIL |
| Number | 11(100%)* | 0(0%) | 12(92%) | 1(8%) | 10(91%) | 1(9%) | 8(88%) | 1(12%) |
| Numeration | 7(64%) | 4(36%) | 11(85%) | 2(15%) | 6(55%) | 5(45%) | 8(88%) | 1(12%) |
| Addition/ Subtraction Concepts | 7(64%) | 4(36%) | 5(38%) | 8(62%) | 6(55%) | 5(45%) | 7(78%) | 2(22%) |

*All numbers and percentage figures indicate number and percent of students showing mastery of the skill. Mastery was defined as answering 75% of the items correctly.

7. The correct score means for addition and subtraction computation (both speed and power) are given in Table XXI for each of the grade/age groups. As shown, younger first graders performed as well as or better than the older first graders for all but the subtraction power test. Older second graders consistently scored better than the younger second graders.

Table XXI

Summary of Computation Mean Scores
For Grade/Age Groups

| OPERATION | COMPUTATION TYPE | GRADE ONE "YOUNG" | GRADE ONE "OLD" | GRADE TWO "YOUNG" | GRADE TWO "OLD" |
|---|---|---|---|---|---|
| Addition | Speed | $\bar{X}$ = 21(70%)* | $\bar{X}$ = 17(57%) | $\bar{X}$ = 23(77%) | $\bar{X}$ = 28(93%) |
| | Power | $\bar{X}$ = 9(75%) | $\bar{X}$ = 9(75%) | $\bar{X}$ = 10(83%) | $\bar{X}$ = 11(92%) |
| Subtraction | Speed | $\bar{X}$ = 15(50%) | $\bar{X}$ = 10(33%) | $\bar{X}$ = 17(57%) | $\bar{X}$ = 25(83%) |
| | Power | $\bar{X}$ = 5(42%) | $\bar{X}$ = 6(50%) | $\bar{X}$ = 8(67%) | $\bar{X}$ = 10(83%) |

*The percentage given is the percent the mean is of the total possible number of items.

Emerging Patterns

A mixture of results seems to have come from the age comparison. Some were expected, others were not. As one would expect, the percentage of children at the concrete operational stage went up with the age of the students, both within one grade and between the two grades. On the other hand, one would not expect that the percentage of children classified as "field independent" would increase with age, yet it did. Items #1 and #2 clearly indicate that field independence was more likely to occur in the "older" children of a grade level.

In performance on the mathematical subtests of the SAT, in the first grade the younger students outperformed the older students; in the second grade the reverse was true. The older students outperformed the younger students. This is in line with existing research, which shows no

consistent differences in the achievement levels of various age groups within a single grade.

From Table XX and item #7, the same mixed pattern of achievement results is evident. There seemed to be no differences between the age groups in performance on the numbers concepts subtest. For the numeration subtest, the older group outperformed the younger group in both the first and the second grade. Finally, for the addition/subtraction concepts, in the first grade the younger group had a higher percentage of children showing mastery of the concepts, while the second grade had a higher percentage of the older group showing mastery of the concepts. It should be noted, however, that across both grades there was a sizeable decrease in the number of students showing mastery of addition/subtraction concepts as opposed to either the number or the numeration test.

Finally, one interesting result is evident in Table XXI. Throughout all other comparisons, the number of students showing mastery of subtraction skills always decreased sharply as compared to addition test results. This was not the case for the older second graders. Their addition and subtraction scores in computation (both speed and power) were quite similar, and this group was the only group to score high on the subtraction computation tests.

### Metropolitan Readiness Test: Quantitative Section

Every participating student was given the Metropolitan Readiness Test (MRT) as a measure of entry behavior. The scores were intended to be used as an indication of areas of potential strength or weakness in mathematics. The students were grouped according to performance on the MRT into "high"

and "low" groups. The high group contained those students with raw scores of 22 through 24 (8th or 9th stanine) while the low group contained those students with scores of 0 to 14 (1st through the 4th stanines). All scores were for the quantitative section only of the MRT. The results were as follows:

1. A larger percentage of field sensitive students scored in stanines 1 through 4 (77%) for the quantitative section of the MRT, as opposed to 79% of the field independent students falling into the 8th or 9th stanine on this section of the MRT.
2. 84% of the MRT high group scored above stanine 7 in the SAT math comprehension section, as contrasted to 21% for the low score group.
3. 79% of the MRT high group scored above stanine 7 in math computation, as opposed to 29% for the low score group.
4. A larger percentage (38%) of the students from MRT high group were found to be at the concrete operational stage, as opposed to 18% of the low group.
5. The majority of students (64%) in the MRT low group were assessed at the preoperational level of cognitive development.
6. For the MRT high score group, the results in the number, numeration and addition/subtraction concepts were as follows: 100% of the students in this group demonstrated mastery of the number concepts, 87% showed mastery of numeration concepts, and 65% exhibited mastery of the addition/subtraction concepts. On the other hand, for the MRT low group, 85% showed mastery of number

concepts, 54% mastery of numeration concepts, and 15% demonstrated mastery of addition/subtraction concepts.

7. Table XXII contains the correct item means for students in the MRT high and low groups obtained for the computation tests in addition and subtraction. As can be seen, there were minimal differences between the groups.

Table XXII

Summary of Computation Mean Scores
For MRT Performance Groups

| OPERATION | COMPUTATION TYPE | MRT GROUP LOW | MRT GROUP HIGH |
|---|---|---|---|
| Addition | Speed | $\overline{X}$ = 22(73%)* | $\overline{X}$ = 23(77%) |
| | Power | $\overline{X}$ = 8(67%) | $\overline{X}$ = 10(83%) |
| Subtraction | Speed | $\overline{X}$ = 16(53%) | $\overline{X}$ = 16(53%) |
| | Power | $\overline{X}$ = 6(50%) | $\overline{X}$ = 7(58%) |

*Percent figures given in parentheses indicate mean score/total score comparisons.

Emerging Patterns

As might be expected, and as shown in items #2 and #3, a high score on the MRT is associated with a high score on the SAT, as far as the quantitative sections of each of these tests is concerned. High performance on the MRT was also associated with high percentages of students showing mastery of the number, numeration and addition/subtraction concepts test. However, for students in both groups there seemed to be a drop in the scores on the addition/subtraction concepts test.

As seen in item #7, computation performance tended to be about the same for both groups. Both groups also achieved lower scores in the subtraction computation tests than in the addition computation tests.

## Language Task Results

During the first year of the study, preliminary tests for children's language ability were devised. The tests were designed to measure children's ability with specific language constructions, especially various phrases and/or concepts which might be involved in the teaching of mathematics. These preliminary results were examined and modified into the language tasks as listed in Chapter Two. During the second year of the study, these modified language tasks were given to all first grade children in the study in an effort to ascertain their skill with the specific tasks.

The data summarizing the children's results on the language tasks are found in Tables XXIV and XXV. It should be remembered that students at School B were English-monolingual students, while students at School C were predominantly Hispanic, with English language skills varying from Spanish-monolingual, to bilingual, to English-monolingual, as measured by the Language Assessment Scales. It should also be noted that only Hispanic students were tested at School C.

Table XXIV

Average Scores on Ten Language Tasks For School B Students

| | A | B | C | D | E | F | G | H | I | J |
|---|---|---|---|---|---|---|---|---|---|---|
| Maximum Possible | 2.00 | 2.00 | 2.00 | 3.00 | 3.00 | 4.00 | 2.00 | 4.00 | 2.00 | 2.00 |
| Mean Achieved | .67 | 1.59 | 1.55 | 2.28 | 2.76 | 3.31 | 1.79 | 2.97 | 1.23 | .93 |
| % of Possible Achieved | 34% | 80% | 78% | 76% | 92% | 83% | 90% | 74% | 62% | 47% |
| Number of Students | 27 | 29 | 29 | 29 | 29 | 29 | 29 | 29 | 30 | 30 |

Key:
A = Biggest Number task
B = Use of "Before"
C = Use of "After"
D = Passive Sentence: Inanimate Agent Animate Object
E = Passive Sentence: Animate Agent Inanimate Object
F = Passive Sentence: Animate Agent Animate Object
G = Passive Sentence: Inanimate Agent Inanimate Object
H = Relativization
I = Mass/count Answers
J = Mass/count Questions

Table XXV

Average Scores on Ten Language Tasks For School C Students

| | A | B | C | D | E | F | G | H | I | J |
|---|---|---|---|---|---|---|---|---|---|---|
| Maximum Possible | 2.00 | 2.00 | 2.00 | 3.00 | 3.00 | 4.00 | 2.00 | 4.00 | 2.00 | 2.00 |
| Mean Achieved | .74 | 1.48 | .96 | 2.04 | 2.17 | 2.66 | 1.48 | 2.30 | .83 | 1.04 |
| % of Possible Achieved | 37% | 74% | 48% | 68% | 72% | 67% | 74% | 58% | 42% | 52% |
| Number of Students | 23 | 23 | 23 | 23 | 23 | 23 | 23 | 23 | 23 | 23 |

Key: A = Biggest Number Task

B = Use of "Before"

C = Use of "After"

D = Passive Sentence:
Inanimate Agent
Animate Object

E = Passive Sentence:
Animate Agent
Inanimate Object

F = Passive Sentence:
Animate Agent
Animate Object

G = Passive Sentence:
Inanimate Agent
Inanimate Object

H = Relativization

I = Mass/count Answers

J = Mass/count Questions

1. Clearly, from an examination of Tables XXIV and XXV, not all the children at either School B or School C had mastered all the language skills. As can also be seen, a greater percentage of children at School B displayed mastery than at School C. This was obvious and should have been expected, as the children at School B were English-monolingual students and those at School C were Hispanic students with varying ability in English.
2. While attempting to differentiate between the physical and numerical qualities of "biggest," students at both schools had difficulty. Only 37% (School B) and 34% (School C) of the students displayed mastery of the two uses of the term. Many children tended to focus on only one dimension of the situation and failed to perceive the term as having a physical and a conceptual dimension.
3. Children at both schools did reasonably well with the concept of "before" as used in a sentence, but again, all students did not achieve the desired answers. 80% of the students at School B were successful, 74% of the students at School C. The difficulty the students had came most often when the order of the actions reversed the order of the words in the sentence. (Ex. Before you pick up the ball, raise your hand.)
4. Results for the use of "after" were similar to the "before" results at School B, but a much lower percentage of students at School C were able to correctly demonstrate the usage. 78% of the students at School B were successful with this term; 48% of those students at School C could correctly demonstrate the actions. Again, the

most difficult sentences were those in which the order of the words was opposite from the order of the actions. (Ex. Raise your hand after you pick up the ball.)

5. For the different versions of the passive sentence construction, children at School B again did slightly better than students at School C. The pattern of responses was the same for both groups. 76% of the students at School B used the first passive (inanimate agent, animate object) correctly; 68% of the students at School C were correct in interpreting the sentences. For the second passive sentence construction (animate agent, inanimate object), the respective figures were 92% and 72%. Both groups had the highest percentage correct figures for these sentence types. For the third type (animate agent, animate object), 83% and 67% of the students at each respective school correctly interpreted the sentences. For the fourth type (inanimate agent and object), 90% and 74% of the students responded correctly.
6. Students' performance with sentences involving relativization was rather weak; 74% of the students at School B correctly interpreted usages of this type and 58% of the students at School C were correct. The obvious error was that when the actor was separated from the action by an inserted clause (Ex. The girls who played with Jim loved football.), children often stated that "Jim loved football." The latter part of the sentence was perceived separately and thus misunderstood.

7. Finally, children's performance on questions dealing with mass and count nouns was low at both schools. 62% of the children at School B could answer the researcher's questions correctly; only 47% could phrase their own questions properly. Students at School C reversed this trend. Only 42% could answer the researcher's questions correctly, while 52% could phrase their own questions properly.

Emerging Patterns

The most important point was that children at neither school displayed mastery of all skills tested. This has two obvious results. First, it was apparent that language skills of first grade children were not totally developed. There were some language structures which caused difficulty. Secondly, many of the difficult language structures were used in math instruction, or were concepts which were part of the math curriculum. Hence, the first grade teacher cannot assume that all children will be interpreting his/her comments in the manner desired.

A problem in differentiation, such as the children had between the two uses of the term "biggest," could easily cause difficulties in the classroom. The teacher who asks the child to name the largest numeral, and who carelessly draws one numeral physically larger than the other on the blackboard, has just caused a problem for well over half the children in a first grade classroom.

The fact that the use of "before" and "after" had not been completely mastered, especially when the order of the actions did not correspond to

the order of the actions in the sentence, may have corresponding difficulties in the mathematics classroom. Teachers are prone to make statements as:

> "After you have written the one's figure in your answer, carry the ten's figure to the next column."

A simple reversal of this statement may cause the child difficulty.

> "Carry the ten's figure to the next column after you have written the one's figure in your answer."

Such a reversal might cause the child to enter a figure in the wrong position. It is obvious that in math, changing the order of the actions may often change the result of the actions. The primary grade teacher must be careful not to use constructions which the child has not mastered. The larger problems of Hispanic students in this area may well stem from lack of familiarity with the language.

For sentences involving the use of passives, a greater number of students understood sentences with an animate agent/inanimate object (Ex. The boy was hit by the car.) than understood sentences where there was an inanimate agent/animate object (The car was hit by the boy.). Similarly, more students were correct when there was an inanimate agent and object than when there was an animate agent and object. Such difficulties may indicate that children rely on semantic cues still at this age. Hence, sentences which are reversible in nature (where there are no semantic cues given) may cause difficulties for the students.

Mistakes in both relativization and passive sentences can ultimately cause difficulty in math instruction. Such sentences as "Three is subtracted from five" or "Eight is divided by three" are both passives of the type where

children had the greatest amount of difficulty. The partition division objective which teachers rated as being very difficult for primary grade students was phrased in the passive form. (Fifteen pieces of candy are divided equally amongst three children.) Relativization is especially easy to do when speaking orally; teachers often insert dependent clauses between the actor and the action. Every time this is done, many children in a first grade classroom become lost.

Finally, children at both schools did poorly on the use of mass and count nouns. The phrases used in describing these nouns in questions such as "how many" and "how much," are often used in word problems. One reason that children have difficulty with word problems may stem from the fact that they have difficulty with the concepts and/or language used with mass and count nouns.

## Summary

Several summary statements can be made; some of them will serve as the basis for developing hypotheses for further study. The reader is reminded that the end products of case study research can only be hypotheses for formal study; no direct conclusions can be made from research of this type. General summary statements, then, include the following:

1. Field independence tends to be associated with achievement in mathematics, while field sensitivity tends to be associated with lower levels of achievement. These general statements seemed to apply to both concept tests and computation tests.
2. Field independent students tended to perform better on the precise thinking exercises, demonstrating a higher ability to use and apply rules.

3. Bilingual Hispanic students tended to perform at higher levels than did English-monolingual or Spanish-monolingual Hispanic students. This was especially true for computation; less so for concepts.
4. Age of a child within a grade did not seem to be related to achievement. One large exception appears to this statement. Eight year old second graders were the only students to do well in the subtraction tests, both computation and concepts.
5. Children's language skills have not completely developed by the first grade; there are several specific usages which occur in math teaching which cause children difficulty.

Chapter Six

# INTERPRETATION OF CASE STUDY DATA

Three main goals included in the purpose of this study centered on the development of math tasks. They were:

1. Data on the development of math concepts in young children.
2. Data on the use of concepts from number and numeration in subsequent mathematical tasks.
3. Data on the difficulties children have with math concepts.

While much of the previous work reported focuses on parts of these topics, the researchers here would like to present a summary of the data under the headings identified. The content of the subsequent pages is outlined in Figure 3.

Figure 3

Placement of Data Under Three General Goals of the Research Study

| GOALS | DEVELOPMENT | USE | DIFFICULTIES |
|---|---|---|---|
| CONTENT AREAS<br>A. NUMBER | Area 1: Data From<br>(1) Case Studies<br>(2) Piagetian Tasks | Area 4: Data From<br>(1) Curricular Survey | Area 7: Data From<br>(1) Observations<br>(2) Piagetian Tasks |
| B. NUMERATION | Area 2: Data From<br>(1) Case Studies<br>(2) Piagetian Tasks | Area 5: Data From<br>(1) Curricular Survey | Area 8: Data From<br>(1) Observations<br>(2) Piagetian Tasks |
| C. ALGORITHMS | Area 3: Data From<br>(1) Case Studies<br>(2) Piagetian Tasks | Area 6: Data From<br>(1) Curricular Survey | Area 9: Data From<br>(1) Observations<br>(2) Piagetian Tasks |

The basic logic of Figure 3 is as follows. The development of concepts was examined through the use of Piagetian tasks, observations of students, and interviews with the students. The developmental ideas, then, are based upon the children's abilities. A similar statement can be made about the description of the difficulties children faced. Scores on test items and sections, as well as difficulties observed in the classroom, are reported and discussed. The use of the concepts in the classroom, however, comes from the curricular tasks required of children. As such, it serves as a bridge between the two other categories. The developmental patterns are described, the uses of the ideas required of children are presented, and then the difficulties shown by children are examined.

## The Development of the Concept of Number

The development of the concept of number is well reviewed in mathematics literature; there is little that we need to add to that discussion. This has been one of the most studied areas in mathematics over the years.

The majority of the participating first grade students were assessed and found to be at the preoperational or transitional Piagetian stage of development. This was despite the fact that they were tested toward the end of the first grade year, as many students approached their seventh birthday. According to the results, 80% of the children tested at School B and approximately the same percentage at School C were functioning at this level. By definition, a preoperational child has a narrow range of operations available to him/her. Consequently, skills such as reversing or associativity are not available to the preoperational first grade child.

It is important to point out that the reasoning of children at the preoperational stage, as well as the way they view and understand the world, is very different from that of an adult. Hence, explanations and problems that seem self-evident and obvious to the adult will be received very differently and often misunderstood by the child. Instruction in many areas of math will be difficult at best, with many concepts being unsuccessfully taught until the child begins to use operations spontaneously. Certain classes of mathematical objectives, such as those with the missing term or the inverse operation, might be difficult for the student.

It was also interesting to note that contrary to what is typically observed, the boys in the sample included in this study were more highly developed than girls. 27% of the first grade boys were classified as performing at the concrete operational level, but only 13% of the girls were so classified.

As would be expected of preoperational children many of these tested had difficulty with the conservation tasks. The most difficult of all was the conservation of length task. At first glance this task would not appear to be as important to the study of number as conservation of amount. But one tool for teaching the concept of number is the number line, and it could be argued that skill with the number line is dependent upon a child's ability to conserve length. The child must relate the two parts of an addition equation with the total as they are pictured on the number line. As conservation of length was the last of the conservation skills to develop, the use of the number line as an explanation device for first grade children may be somewhat premature.

It is interesting to note that conservation of area seemed to be the easiest of the conservation tasks for the children, but learning about two-dimensional space does not receive much attention in the curriculum at the first grade level. Perhaps it should be given more attention.

With respect to the ordinal number, the students participating in the study were given a seriation task. In this task, the students were asked to arrange a set of twelve straws of differing lengths. First grade pupils were grouped according to high, average, and low ability levels, as evaluated by their teachers. Results indicated that all of the students, regardless of the group, were able to accomplish the task. Students from the higher group were able to finish the task faster than students from lower groups. High ability students were able to arrange the straws in a more expedient manner without resorting to trial and error strategies. Students from low ability groups had to take additional time and trials to find what came "after" the straw that was on the table. Another finding showed that the order in which the straws were placed (i.e., larger on the right, smaller on the left; or larger on the left, smaller on the right) had nothing to do with the ability of the student to complete the task successfully.

It should be noted here that teachers did not rate the ordinal number objectives as being difficult for their children. A rather low percentage of teachers chose the "difficult" categories on this item on the teacher survey for their responses. It would appear that the use of ordinal number was not as difficult for the first graders as previously predicted in the literature.

In summary, most of the students seemed to have acquired the idea of number in the ordinal sense. They would be identified as performing at stage 2 or stage 3 (Copeland, 1974). This is despite the fact that the majority of the students were rated at the preoperational or transitional level of cognitive development. It would appear that ordinal number is not directly related to the child's stage of development.

## The Development of the Concept of Numeration

Several tests were given in an effort to determine the child's skill with numeration and/or place value. The first task was a grouping exercise. Children were given a group of thirty-four objects; they were told that it was necessary to count the objects, but no one wished to spend the time counting each item separately. So, the students were asked to group the objects so that a count could be made just by "looking at" the groups.

Only two children tested made groups of ten for "counting" the set. The remaining children ranged from not being able to make a group at all to making groups of whatever number seemed convenient to them. They almost invariably stayed away from making groups of ten. Some children even made groups of varying sizes. One might conclude that the practice of grouping objects by ten with a "left-over" stack is not ingrained in children, even those children in the second grade who had received a great deal of place value instruction. Copeland (1979) suggests that most children under the age of nine will not make groups of ten. It would be hypothesized, however, that if children relied upon the instruction they received in place value, they would make groups of ten for ease of recognizing and writing the numeral.

The second test given was the place value task reported by Underwood (1977). It was hypothesized that children do not have "conservation of place value," as they often did not recognize a group of ten plus four as having the same number as a set of fourteen separate items. In other words, children could not conserve the amount in the group of ten when the group was separated into individual items.

The task was given to children in both the first and second grades. The children were shown a group of ten sticks bound by a rubber band, with four additional sticks placed nearby. While the child watched, the group of ten was unbound and placed individually before him/her. When asked about the amount of objects, second graders were consistent in stating that there were the same number of sticks before and after the transformation. Those who made this judgement were quite emphatic that "ten was ten, regardless of how you moved it around." A few second grade children, however, did select the unbound sticks as being "more" than there were previously.

For first grade children the results were definitely mixed. Some children were quite confident that a group of ten was the same as ten individual items. All of the children classified as being "high achievers" by their teacher made the judgement correctly. For students identified as being in the "middle" or "lower" groups, however, performance fell off dramatically. One child stated the correct numerical relationship; everyone else tested felt that the amount had changed. Interestingly enough, the children were split evenly as to the fact that sometimes there were "more" and sometimes there were "fewer" sticks after the group was separated.

These results indicated that some of the students were not at a stage of cognitive development where they could understand the concept of place value. One can certainly question the wisdom of the teacher attempting to teach these children place value skills using the traditional approach.

Another skill identified as necessary for the development of numeration concepts was that of class inclusion. The child must be able to identify the "sameness" of sets having similar amounts and be able to put them into the numerical class where they fit.

On the general Piagetian tasks of classification and class inclusion, the performance of children was at the preoperational level or the transitional level for the majority of students in School B and C. The performance of the children on the classification task seemed to indicate that they had a rather limited ability to deal with separating and arranging the dimensions of objects systematically. Even more important was the finding that most of the children exhibited clear preoperational performance on the class inclusion tasks. Children seemed to be unable to consider the fact that objects might belong to different classifications simultaneously and that groups could be merged or separated without the number of items changing overall. These abilities are clearly related to the place value objectives, where the child is expected to complete regrouping actions as the situation demands.

Another of the class inclusion tasks developed by Piaget was the "family relations" task described earlier in this report. The tasks included in this test first ask that the child identify various relationships

which exist, then name the numerical amount of various classes, some in which he must count himself and some in which he does not count himself. (For example: "How many brothers do you have?" versus "How many brothers are there in your family?")

For students identified as high achievers by their teachers, the results were generally mixed. While most could identify and describe some aspects of the words "family," "uncle" or "brother," they could not always give complete descriptions. For the class inclusion/exclusion questions, most of the high achieving students could correctly "count" the number of people in each set and correctly place themselves into and out of the sets as required. The students identified as "low achievers" by their teacher were not able to respond to the first set of questions even when prodded. On the second set of questions, they consistently included themselves when all questions were being posed.

To probe further into the level at which students might have been operating in the area of class inclusion, a last question was given. The question asked the following:

1. There are three brothers in a family. David, Danny and Mike.
    a. How many brothers has David?
    b. How many brothers has Danny?
    c. How many brothers has Mike?

Low achievers consistently included all children in answering the questions. They answered three to all questions. Low achievers did not seem capable of making the class inclusion/exclusion judgements.

The results suggest that those students underachieving were at a stage of cognitive development in which the numeration concepts were not understood.

It seems as if whatever knowledge they had concerning numeration had been learned through "rules" presented by the teacher for their memorization.

## Development of Basic Computational Operations

As there is some evidence that Piagetian tests cannot be utilized as general predictors of achievement (Mpiangu and Gentile, 1975), specific tests were developed or taken from the literature for specific math concepts. These tests dealt with the content covered by objectives identified as "difficult" by half or more of the teachers responding to the teacher survey. Since these objectives were termed difficult for students to learn, it was hypothesized that a developmental difficulty might be at work for those particular concepts or operations.

Specific developmental tasks for the concepts and accompanying results were organized by operation.

### Addition

Since teachers identified the definition of each operation as a problem, tests dealing with the definition of addition were given. The first test for addition was the addition of classes task. The child was given nine blocks. Seven were blue, two were brown, and all were made of wood. The question posed was: are there more blue blocks or are there more wooden blocks? Most children in the first grade stated that there were more blue blocks; only two students answered correctly. Most children did not add the two colors to form the class. For students in the second grade, the addition of classes task posed no problem whatsoever. All children tested gave the right answer. The closest anyone came to a

mistake was when one child gave a wrong answer, then quickly asked, "What did you say?" When the question was repeated, he changed his wrong response to the correct answer.

This would imply that while children in the first grade had some difficulties conceptualizing the operation of addition, by the second grade they had no difficulties whatsoever. The fact that only two top students in the first grade obtained the right answer for the test indicated that most children did not have this concept in the first grade.

A second test (Copeland, 1974) dealing with addition was also given. This test asked that the child equate the idea of four plus four as being the same as one plus seven.

In the first grade, children consistently made an error when asked to equate these two equations. The one student who achieved the correct answer (a high achiever) first thought one plus seven was more, but then changed her mind and said, "No, it's the same." All other students tested in the first grade stated that one plus seven was different from four plus four. Some said one plus seven was more, some said less, but no one thought it was the same. In the second grade at School A, performance on this task did not improve. Children in the second grade at this school also thought that there was a difference in the totals of four plus four and one plus seven. Almost without exception, the second graders chose one plus seven as having the greater amount.

An interesting sidelight was the definite nature of some of the children's answers. They had no doubt about their responses. When questioned again, they repeated their answers; often those repetitions were quite forceful.

Another point should be made. In the second grade classroom prior to and during the period of the addition task testing, the children were repeatedly doing classroom drills with differing ways of stating problems equal to a given number. The teacher would write a numeral on the board, for example, 15, and the children in the class were expected to state various number sentences equal to the numeral. The children would give numeral sentences as 6 + 9 = 15 or 8 + 7 = 15, etc. The class had become quite proficient with this task, especially in recognizing commutative reversals. Yet they still had difficulty with the Piagetian task.

One should be cautious in generalizing about the second grade results, as most of the students who could not perform the task came from School A, where achievement levels were definitely lower. Second graders at School B were more proficient with the task. Almost all second graders at School B answered the questions correctly.

A general hypothesis might be that children are learning the concepts of addition at a stage in which they are not really understanding the ideas as much as they are memorizing the number facts. This would be in agreement with Piaget's interpretation that many children are indulging in rote learning when learning concepts for which they have no conceptual understanding.

Multiplication

The concept of multiplication was also rated as difficult by teachers responding to the teacher survey, especially those objectives dealing with identification of all possible combinations in a Cartesian cross of two sets. Three tasks were given to examine children's reasoning capabilities with this particular concept.

The first was Piaget's combinations task. There were four parts to this task. The child was given in order: (a) a set of four blocks of differing colors and asked to make all possible groups of two; (b) a set of four blocks of differing colors, and asked to make all possible groups of three; (c) a set of five different colored blocks and asked to make all possible groups of two; and (d) a set of five colored blocks and asked to make all possible groups of three.

When first graders were given a set of four objects and asked to find groups of two, children used a random procedure, simply matching one color with another until they thought they had achieved all the possibilities. They usually had. The same was true when first grade children were given a set of four blocks and were asked to make groups of three. Most children used the element that was not in the set of three as their check point; when all four colors appeared in the "omitted" position, they were through. These results were true for students from all schools in the study.

When solving both of these exercises, however, the first graders tested showed no logical, coherent way of organizing the blocks in order to make all the possible pairings. A completely random approach was utilized; they took the blocks in any haphazard manner to make combinations until they could find no more. In most cases they had achieved all the possible combinations when a set of four blocks was given. That was not the case for a set of five blocks.

When first grade children were asked to make groups of two from a set of five, their ability to solve the problem decreased immensely. Again, they used a random procedure. Two blocks were simply picked out of the

set, then another two, and another two, etc., until the child said he was done. There was no systematic procedure used to determine if all groups had been made. Consequently, all children missed some of the possible groups. For the task requiring groups of three from a set of five, most first grade children made two or three groups, became frustrated, and quit. Quite obviously, the task was beyond them.

The lack of a logical procedure to follow meant that the larger set presented problems larger than the random procedure would solve. Children seemed to "intuitively know" when they had made all possible combinations from the small set; the addition of one element, however, raised the number of possible combinations past the ability of the child to "picture" the total number of possibilities.

We would hypothesize, then, that introducing the idea of combinations in a first grade class is beyond the student's conceptual ability. While combinations per se are not taught, the math systems program does include objectives which state "The child will identify all possible combinations from a Cartesian cross, possibilities less than twenty-five." Teachers identified this as a difficult objective. There seems to be little cause as to wonder why; first grade children lack the logical thinking skills necessary to make all of the various combinations from the crossing of two sets.

For second graders, the same general results were true for the majority of children at School A. They would randomly solve the problem asking for groups of two from a set of four. They used the same procedures while making groups of three from a set of four, usually achieving all the

solutions. But when given a set of five, the random procedure no longer worked; they could not arrive at all of the various combinations.

Students at School B, displayed a slightly different pattern. The majority of children could still not achieve all the combinations. They still used the random procedure, which worked only with the smaller sets but did not give them the solution for larger sets. Two children, however, differed from this procedure. They first attempted to solve the problem in a random way. However, both children said upon the completion of the task, "Wait! I know a better way of solving it." The researcher then allowed them to do the task a second time. During this second time through, they used a logical problem solving structure. They would take the first color and pair it with all remaining colors. Then they would use the second color and pair it with all remaining colors. In this manner, they finished the exercise. One additional child discovered this procedure, but made an error in her problem solving strategy. She put the first block at the end of the line and continued to use it, then put the second block at the end of the line and continued to use it, and so on, thus creating combinations previously made. Consequently, she never arrived at a point where she knew she was finished. She had the idea, but could not quite complete the final reasoning step.

A similar test was given using the concept of permutations, but it was too difficult for all children involved. Children were asked to make all possible permutations (an example replaced the term) from a set of four. No child in any grade was able to make all the permutations from a set of four, so no further questions were given. This concept appeared to be

much more difficult than the combinations task for all students tested. It was dropped from the test series.

The third multiplication task given was an exact test of the objective included in the math systems program. The children were asked to make all possible elements in a Cartesian product formed from two sets. There were two tasks included. Since children had solved the combinations task with smaller sets, small sets were used in the first Cartesian product task. Children were asked to prepare all possible elements of a Cartesian product using a set of two triangles and a set of three squares. The second task used larger numbers; the children were asked to pair elements from sets of three and four.

A rather interesting phenomenon occurred. Two high-ability, first grade children at School C were able to use a logical and sequential procedure for solving the problem. All other children were not able to arrive at a logical problem solving strategy. At School B, this was also the case. A high achieving first grade student made all the possible combinations from the two sets in both tasks that he was given. No other first grader passed both tasks. No second grader passed all the tasks.

There was evidence from second grade children at School A of a pattern similar to that found for the combinations test. Some children achieved, through a random procedure, all of the possible combinations of elements from a set of three and a set of two. Yet no second grade child achieved all possible combinations when a set of three was crossed with a set of four. Results at School B were similar.

As stated earlier, teachers identified the objective where children made combinations out of two different sets as being a difficult one. This would certainly appear to be the case from our test results. Children did not have the logical thinking structures available to them for solving the problem. Even when given examples of a logical problem solving procedure, most first and second graders did not achieve the desired procedure. One would argue from a developmental point of view that the cognitive structures necessary to solve the problems were not there. Therefore, despite any amount of teaching or hinting or prompting, the children would continue to use the thought patterns available to them to solve the problem.

Missing Term Problems

The teacher survey also identified missing term objectives as causing a great deal of difficulty for students. Children in the selected classes were given two tests for this particular concept, both dealing with missing terms in an addition problem.

In one task, the children were given a row of blocks with seven items in it and beside it another row of three blocks were placed. They were then asked how many more blocks would be necessary to make the seond row have as many as the first row. Second graders were quite adamant in stating that four was the answer. They had little or no difficulty with the concept.

Many first grade children, however, could not successfully solve the task. About half of the students tested could not place four blocks in the row to achieve the solution.

In a similar, but probably more difficult, exercise children were asked to identify the correct number sentence for a story problem. The story problem was of the missing addend type. Twenty-seven of the twenty-eight students present that day at School C, could not select the correct number sentence from the four possible items.

During the first year of testing, students at School A were tested for the missing addend problem with a series of two tests. First, they were given blocks. A row of five blocks was made and a row of two blocks was placed beside it. Children were asked: "What do you have to put with the two blocks to make that set have as many as the set of five blocks?" An interesting thing happened. Many children matched the blocks in a one-to-one correspondence. If the blocks were placed side by side, they could obtain the answer of three necessary for the equation: $2 + __ = 5$. However, if the experimenter placed the blocks in an order other than the one-to-one correspondence, some children had difficulty with the problem. For example, the experimenter placed eight blocks in a row and then 3 blocks in another row, beside the row of eight. However, the 3 blocks were placed closely together, so that they occupied the same space as 2 blocks in the row of eight. Then the children were asked how many more were necessary to complete the number equation: $3 + __ = 8$. Invariably, the children using the one-to-one correspondence procedure continued to use this strategy, so they placed six blocks in a one-to-one correspondence and achieved the wrong answer. This would indicate that while the children have achieved a solution process, usually giving them the right answer, they are unable to modify it in any way. This would certainly indicate that they did not understand the concept involved in the missing term problem with addition.

## The Use of Number in Mathematical Tasks

The data looked to by the researchers for information on the use of number and numeration concepts in subsequent mathematics learning were included in the content analysis. Since all objectives from subsequent levels of the systems program were identified and grouped according to topic, it was felt that this data would include excellent sources for considering the question, "How do children use number and numeration concepts in arithmetic?"

In order to define the use of number concepts in primary grade mathematics exercises, we first must divide the concept of number into two parts: ordinal and cardinal uses of number. These two tasks are separate; they provide unique and different aspects of number for the child to focus on.

### Cardinal Use of Number

There are many uses of the cardinal principle of number in mathematics. One of the most early uses of the cardinal principle is for the child to make comparisons between the amounts of various sets. Many of these comparisons also require the use of a vocabulary of rather precise terminology. The child must make larger/smaller comparisons, more/fewer comparisons, or equal/unequal comparisons. He/she also must deal with similar terms concerning the absolute amount in a set. Such ideas would include the use of fewest, least, greatest, or most. In every instance the child is concerned with the comparison of amount, using the one-to-one correspondence as his/her basic strategy for determining whether sets are equal or unequal.

An extension of this usage lies in comparing sets on the basis of "one more" or "one less" element. Such an objective underlies the skill of counting, or more correctly, seeing the logic of the counting numerals. These objectives are frequently presented in the first and second grade mathematics curriculum.

A second major use of number becomes evident when the child learns the relationship between number and numeral. Objectives dealing with this topic are almost always ordered by the size of the numeral; there is ample research to indicate that children move up through the Piagetian stages according to the size of the numbers they are asked to deal with. Consequently, the objectives first ask the child to equate number and numeral for sets less than five, moving on to sets of five to ten, and subsequently up through the larger numerals.

A similar use of number is encountered when the child equates place value notation to sets of ten with n objects left over, where n is 9 or less. Underwood (1977) has described this as the conservation of place value; the required concept is one of equating two sets, one with ten elements grouped together and one with the ten elements spread apart. While this is just an extension of the number-numeral relationship, it does move on into larger numbers and adds one new idea, that of the conservation of place value.

The concept of number is used to teach the child two main ideas. First, he/she is asked to make comparisons between amounts; the basic skill is that of beginning to consider the n of a set as the important variable. The second major skill is that of learning that numerals can be used to

represent those amounts. Very quickly in the primary grade curriculum the child moves from the use of number into the use of numeration systems as symbols for number. Most of the subsequent operations the child learns to perform are really operations upon numerals, not numbers.

There is one exception to this statement. The concept of number is used in defining all subsequent mathematical operations. Given a number pair, such as 9, 3, the child is asked to recognize that four different answers are possible for that number pair. He/she may achieve answers of 12, 6, 27, or 3 for the number pair, depending upon which operations he/she performs upon the numerals.

As a result, the child must understand the nature of the operation he/she is performing upon the number pair in order to achieve the different results. The curriculum resorts to the concept of number as the means for representing each of these operations. For example, set union is given as a definition for addition; subtraction is pictured through the removal of a subset. (Technically, it's a take away situation that is shown in this manner.) Repeated addition and Cartesian products are offered as definitions of multiplication. And finally, the partitioning of sets and repeated subtraction are offered as definitions for division. All of these are performed with sets, so the concept of number is used while defining each of the four basic operations.

Use of Ordinal Number

Two verbs summarize the use of ordinal number in the primary grades. The child is asked to "match" ordinals to objects in positional order; he/she is also asked to "label" the ordinal position of given events or

pictures. In other words, the child is expected to learn the various ordinal positions and then to apply that concept.

The application of the concept centers on such ideas as recognizing that page numbers are an ordinal use of number. The child is asked to apply numbers to positions in order to count sets consistently throughout the elementary school curriculum. But there are few additional mathematical objectives which deal solely with the ordinal concept of number. This knowledge is simply assumed in order that the child may complete subsequent mathematical tasks which require the skill. (For example, Piaget argues that counting a set presupposes the fact that the order arrangement of the set does not matter. He feels that the cardinal and ordinal principles are related and are really inseparable.)

## The Use of Numeration Skills in the Primary Grade Mathematics Curriculum

Items such as reading and writing numerals, matching the numeral to a set containing the number, and making comparisons between numerals using the greater than/less than concept are the first numeration objectives the child encounters. All of these might be termed basic skills of numeration. They are the components involved in learning the concept of numeral; the application of the concept comes later in the curriculum.

An application may be found in asking the child to extend the concept of numeral into the place value system. Early objectives call for skill in grouping; the student is asked to identify N sets of M objects. This may be done with or without remainders. One can readily see the idea: the child is being asked to make repeating groups of the same amount.

He/she then extends this concept into the place value system by making groups of ten with some numeral left over. He/she is, in essence, learning the value of the first two positions in the place value system. The often repeated exercises in the primary grade curriculum deal with the numeral in the ten's "house" and the numeral in the one's "house." It may be an interesting way of describing the topic for a first grade child, but it does describe the concept of what numeral placement means in determining amount.

The child then moves on to expressing the place value position in terms of expanded notation. For a great portion of the curriculum, the major numeration objectives deal with extending the place value system throughout two, three, four and five places as the child moves up in the sizes of numerals he/she works with. The first two or three of these levels are always expressed in expanded notation.

Finally, the child begins to use the place value system. First, the skill of regrouping in both addition and subtraction is utilized; regrouping is basically a numeration skill. The child must recognize when an answer needs to be changed from one place value position to another. Similarly, multiplication uses the concept of regrouping and hence requires the skill of numeration. But multiplication introduces a new skill with numeration. In problems involving two two-digit numbers the second of the two products must be moved over one place value position. For many children, this is a difficult concept; it rarely appears, however, in the primary grade curriculum.

Numeration is definitely required while solving division problems. The place value of the quotient figure must be carefully handled. A standard error is the omission of a zero in the quotient; the figures are positioned in a way that it is impossible to determine their place value position. Consequently, the student does not quickly recognize that one numeral is missing and a zero must be entered in his answer.

The skills of numeration extend into the higher grades also. The place value system is extended into the decimal system in the fifth or sixth grade. The use of decimal notation is quickly compared to percentages. It can be hypothesized, that many of the problems students have in the subsequent years in the mathematics curriculum are derived from difficulties encountered in numeration. If the child does not master numeration (i.e., place value) skills, he may constantly be in trouble in performing higher level mathematical tasks.

## The Use of the Basic Algorithms in Subsequent Mathematical Learning

Objectives dealing with the basic algorithms can be categorized as beginning with the definition of the task and finishing with its application. A basic series of objectives is outlined for the teaching of all four basic algorithms. Generally, the following steps are followed: 1. The operation is defined, usually with sets; 2. Word problems or picture expressions of the operation are examined in order to translate them to number sentences; 3. Basic facts are memorized, including the reversal of these basic fact expressions in a missing term problem; 4. Practice is given in working the steps of the algorithm.

After the student has mastered the algorithm, he/she moves on to its application. Applications tend to fall into three general areas. First of all, the child is asked to solve word problems. Word problems attempt to picture real life applications of mathematics; this is more or less true depending upon the content of the word problem. To summarize the child's experience with word problems, he/she is expected to find number situations which apply an algorithm in life-like situations.

The second use of an algorithm lies in solving problems dealing with money. A great deal of the application period is spent on the use of any algorithm with dollars and cents. The reason is obvious; most of the numerical situations that a child will deal with in his life stem from money problems.

The final use of any given algorithm lies in the use of that algorithm as part of a subsequent algorithm. For example, every multiplication problem of two digits or more depends upon addition as the means of solving that algorithm. Division uses both subtraction and multiplication; without these skills the child cannot solve the division problem. Consequently, one major use of early mathematical algorithms is as a part of a subsequent algorithm.

One might hypothesize that the operation of addition may not be mastered in the first or second grade. It will be practiced repeatedly in the third and fourth grades as the child masters the operation of multiplication. Likewise, subtraction will be practiced repeatedly in the fourth and fifth grades as the child masters the operation of division. These subsequent practice and review sessions may finalize the child's skills in each algorithm.

## Children's Difficulties With Number Concepts

The final three sections of this chapter deal with difficulties children encountered in learning the concepts included in number, numeration and the basic algorithms. No effort has been made to catalog all difficulties; in such areas as the basic mathematical operations, the categorizing and cataloging of mistakes has been done widely in the past. Rather, an effort has been made to note the special difficulties encountered by children in this sample.

This first section deals with problems identified in children's learning of number concepts. Teachers, responding to the teacher survey, did not identify any number objectives as being difficult for their students. (More correctly, such objectives were not identified as "difficult" by over half of the responding teachers.) To the contrary, many of the number objectives were perceived by teachers as being "easy" for their students.

Only when the elements of a set had to have some operation performed upon them did teachers rate the objective as being difficult. For example, the objective requiring the child to partition a set into a given number of subsets was perceived as difficult by a majority of teachers. But no objective which required the child match elements in a one-to-one correspondence or to compare amounts was perceived as difficult and rated accordingly.

When working with the children in the selected classrooms, one immediate fact impressed the researchers. It seemed that at the beginning of the first grade year, just as the children were beginning their educational careers, that those children who would succeed and those children who would fail

could already be identified. The background the child brought with him/her to the first grade had already determined his/her performance. This was especially true in mathematics. The most successful children brought a knowledge of number with them to school. They knew how to count; they could compare sets by number. Most successful children had number/numeral correspondence for numerals under 10. The child who did not have these skills was very apparent.

This evidenced itself in a variety of ways. One was the child's knowledge of and willingness to use the concepts of number/numeral. For those children who would learn in the subsequent months, if the researchers held up three fingers, those children would say three. The children who would face numerous failures in the future months would not respond, or if they did, they would question their response and/or often respond incorrectly.

When given simple number sentences to solve (2 + 1 = __), the successful children immediately began to count it out on their fingers or to use the counters provided by the teacher. They knew to hold up 2 fingers on one hand and one finger on the other hand and then to count them. They knew the meaning of the numerals. The unsuccessful child never saw this relationship. If the researcher held up 2 + 1 fingers, the unsuccessful child could be led to count them. But for the next problem, it still never occurred to him/her to represent the numerals with his fingers.

A subsequent and often observed behavior amongst the lower ability students was persevering. This seemed to be the child's defense against the fact that he was not certain of what to do. In one classroom, the researcher observed one child write his name on the paper, erase it, try

again, erase it and tear the paper, get out a new sheet of paper, and begin the entire process over again. Many students completed the assignment before this child even began. Perseveration severely limited the actual amount of work done by many children; part of the learning problem these children had was associated with the fact that they never completed enough practice exercises to finalize the learning they did manage.

Much more importantly, however, was the question: Why did the child persevere? One consideration must be that the child was asked to do exercises which he did not comprehend. Repeated examples and illustrations, given to the child in a one-to-one relationship with the tutor/researcher, failed to elicit any spontaneous generation of the procedure to use for the next problem. One conclusion might be that the child had no idea of what to do: as long as an adult would demonstrate the reasoning for him/her and tell him/her what to do, he/she would do it. But there seemed to be no understanding, no willingness to try the next problem "on his/her own."

Some of the Hispanic students suffered problems in concepts related to number which were definitely language related. The discrimination between the terms (in English) fewest, least, greatest, greater than or less than caused some students difficulty. The problems in mastering these concepts seemed to be language problems; once the concepts were presented in Spanish and a relationship was established between the English and Spanish labels, the students were able to demonstrate progress in the mastery of these relationships.

For a small group of students, difficulties in learning concepts related to number were found to be of a nature apart from language interference. When instruction and activities were carried out in the student's

first language (Spanish), the child was still unable to do the exercises and problems. The concepts were:

a. Identification of sets with the fewest/least/greatest number of elements.

b. Use of less than or greater than to describe order relations between pairs.

c. Identification of sets having one more (or one less) element than another set, all numerals ten or less.

These children could count the set, but when asked to show the relationship between two sets, they could not indicate this to the teacher or the researchers. The skill of counting had no relationship to comparing sets or describing numerical relationships. All of these children were assessed as preoperational; they appeared to follow the classic description of the preoperational child and his/her mathematics abilities.

For the majority of Spanish-speaking students, the concepts of number did not pose a serious problem. 75% of those students tested displayed mastery of the numeration concepts, in spite of the fact that instruction was provided in English, a language in which some students were not proficient. It can be hypothesized that the concepts involved in number are not as language dependent as others. This might be due to the fact that teaching approaches used for number concepts were basically manipulative in nature. The students worked with blocks and other objects in an effort to visualize the idea of set equality, the one-to-one correspondence, or set comparisons. Such an approach would be relatively "language-free," as compared to instruction in a topic such as numeration.

Further evidence of children's ability with number concepts was given in the criterion reference test administered at the end of the project. The mean scores for this section of the test were quite high, both for first and second graders. After defining mastery as achieving 75% of the test items correctly, 100% of the first graders achieved this level and 91% of the second graders.

In addition, when the performance of students was classified by the various Piagetian stages, 100% of those students classified as concrete operational or transitional mastered the skills in the number section of the criterion referenced test. Even 81% of those classified as preoperational exhibited mastery of the concepts. Similar percentages were obtained for the classifications field sensitive (85%) and field independent (93%).

In conclusion, the lack of problems in the area of number might be due to the fact that number concepts are taught at the kindergarten level, so the majority of first grade children have seen the concepts previously. Also, many of the children came to school with a good background in number. For those students having difficulty, the concepts definitely appeared to be taught at a time which was "too early." All children showing failure on number concepts were rated as preoperational; all responded with typical preoperational behavior. When given instruction, they failed to generalize the ideas taught to them.

## Children's Difficulties With Numeration Concepts

As with number concepts, the topics included under numeration caused few difficulties for students, as long as the numerals used in the exercises were kept small. The larger the numeral used, the more likely it was to

cause difficulty. Results of all tests indicated that children had more problems with numerals over ten than with numerals less than ten. But by the end of the first grade year, most students displayed mastery of numeration concepts tested for in the numeration subtest of the ABST.

In the teacher survey only one area under the heading of numeration was perceived as difficult by a majority of responding teachers. That cluster of four objectives dealt with the use of expanded notation in the primary grades. Children had difficulty with the use of expanded notation, especially in the regrouping steps required for addition and subtraction.

From the classroom observations, additional difficulties surfaced. When presented with Underwood's (1977) conservation of place value task, many first grade students had difficulty conserving the amount as required in this task. Most students, except those identified as high achievers, felt that there were "more" or "fewer" sticks in the ungrouped format. This would indicate that children would have some difficulty with objectives requiring the use of place value and expanded notation, exactly as teachers reported. One might hypothesize that place value instruction in the first grade is somewhat questionable; children might have difficulty dealing with the very explanations the teacher uses to illustrate the concept of place value.

Another problem surfaced for one of the earliest numeration activities. Many children had memorized the counting numerals when they came to school. In an effort to add meaning to the memorized terms, children were given such exercises as number/numeral matching, order arrangement activities (of the type "two is one more than one," "three is one more than two," "four is one more than three," etc.), and finally exercises which dealt with the

numeral coming "before" or "after" a given numeral. Questions posed would be similar to:

What numeral comes before 6?

What numeral comes after 9?

Exercises of this type, dependent upon the use of "before" and "after," caused a great deal of difficulty in the first grade classrooms, especially those in which the majority of students were Spanish-speaking.

From the language testing done, it was evident that not all children were capable of dealing with activity sequences which included use of the terms "before" and "after." Dealing with a sequence of numerals, and using these two terms to describe the relationships between numerals, must also pose some language difficulties for children. One hypothesis on the difficulty of the terms "before" and "after" might be that children (especially children working in a second language) might not be fully aware of the meaning and implications of the two terms and the related language structures. This problem, then, would be a language development problem; the math concepts should not be presented if the linguistic ability has not yet developed.

In reviewing data from tests given to students, most students showed excellent numeration knowledge at the end of the first and second grades. An item analysis indicated that the majority of the errors appeared for test items dealing with two digit numerals or expanded notation for two digit numerals. The majority of students still achieved the desired percentage (75%) necessary to show mastery of the skill.

On the numeration test, figures ranging from 83% of the concrete operational students to 63% of the preoperational students showed mastery of

the concepts included in the numeration test. When students were grouped according to cognitive style, field sensitive and field independent students performed similarly on the numeration test. 75% of the FS students and 64% of the FI students showed mastery of the concepts included in the numeration subtest.

The only evidence of a problem group of students obtained from the test data was a group of Spanish monolingual students. This group showed the lowest percentage of students mastering the subtest. It appears from this result that the teaching of numeration may rely heavily upon language. Therefore, the Spanish monolingual students either did not acquire the concept or did not acquire the English vocabulary to show mastery of the concepts. This is an area of mathematics where teaching the content in the native language of the student might be helpful.

## Problems With the Basic Algorithms

From the teacher survey, five general areas of difficulty were identified which dealt with children's learning of numerical algorithms. Those areas were (1) use of the number line, (2) problems involving missing terms, (3) use of inverse operations, (4) operation definitions, and (5) operation translations. These will be examined in terms of their developmental difficulty.

One cluster of objectives perceived as difficult by the teachers surveyed dealt with utilization of the number line. The children's difficulty in dealing with the number line can be predicted and explained by the fact that many children in the first grade do not show the ability to conserve length. The use of the number line requires that children understand

thoroughly how lines can be combined, rearranged, and divided and how these lines relate to each other before and after operations are applied. It stands to reason that children who are at an earlier level of cognitive development than the children in the schools tested will have even more difficulty with the number line. This seems to be the case where instructional aid, a tool intended to make understanding of math concepts easier, turns out to be as difficult and probably more so than the concept it was supposed to simplify.

The next two difficult areas identified by teachers can be grouped together. Missing term objectives and inverse operation objectives seem to share the same developmental problems. Both types of objectives require that the child have available to him the operation of reversing. The child must reverse an action mentally so that the situation can be compared before and after the operation. Operations first appear in the child's reasoning during the concrete operational stage. Since the majority of children in the two schools tested were in the transitional stage or the preoperational stage, the ability to reverse was not likely to be available to them. In fact, the children's inability to justify and defend their correct judgements when doing a conservation task reflected partially their inability to use reversibility in order to explain their judgements. As a logical result, objectives dealing with reversals would pose major problems for the children.

The operation translation and definition objectives might also be grouped together, as they present similar developmental problems. Children may have difficulties with these tasks because the objectives require a

clear understanding of the hierarchical nature and the multiple possibilities of class inclusion. Since many of the children in both schools did not understand class inclusion and classification completely, it may have been difficult for them to grasp the rearrangement of groups into new groups that maintain some of the qualities of the old but also have new qualities of their own. Until a child understands the multiple possibilities of class inclusion, he/she will not be able to see the similarities between the new groups and the old groups.

While it does not deal directly with difficulties, it is interesting to examine those objectives deemed "easy" by the majority of teachers responding to the teacher survey. Those objectives were exactly the ones which do not require operations. When examining these objectives, we see that they require mostly the creating of a one-to-one relationship between objects and numerals, the skill of associating numerals with number, or the mastery of a skill that can be learned by rote means. Such skills do not require the understanding of complex ideas or the use of mental operations.

Another developmental difficulty appears to be present for children with the operation of subtraction. Test scores from participating students were grouped by grade. Within each grade, additional groupings were made on the basis of age. For example, for grade one, students from ages six to six and one half years old were classified as "young," and students from ages seven to seven and one half years old were labeled as "old." Similar procedures were used for the second grade. In examining their performance in computation in both addition and subtraction for the speed and power subtests of the Arithmetic Basic Skills Test, a marked

increase in subtraction computation mean scores was found for the "older" second grade students. The "older" second grade students were the only group where subtraction performance equaled addition performance. There was a decrease in performance in the first grade between addition and subtraction computation for both age groups; the differential mean performance between addition and subtraction computation appears to be minimal for the "older" second graders. A clear-cut trend can be seen in performance between addition and subtraction. The older the children were, the less difficulty they found with the operation of subtraction. Children in the second grade tend to do better the older they are. For example, only fifty-seven percent of the "young" second graders displayed mastery of subtraction speed computation, as opposed to eighty-three percent of the "old" second graders. Such an improvement in performance was not noted in the first grade. It would be safe to hypothesize that the children have reached a developmental stage at which they understand the concept of subtraction and can follow meaningfully the algorithm of subtraction.

In summary, one impression that emerges from a comparison of the developmental data to the mathematics curriculum is that in the first grade the curriculum is probably sound from a mathematical, logical point of view. It is clear that attempts have been made to introduce ideas in a logically sound manner and in a way which appears to be meaningful in the scope of the program. In doing so, the meaningfulness of concepts should emerge for the child. The criticism of this approach is that the best logical way to present material may not be the best psychological way in which to do so. The assumption that the internal structure of subject

matter material is the one that will be duplicated in the learner's mind and should be utilized in teaching the material is not necessarily a sound one, even for adults who have the necessary developmental tools to capture the internal structure. Such an assumption is probably a very poor one when dealing with children whose cognitive structure is not fully developed; these children's thinking seems to have a "logic" of its own that is clearly different from that of adults or from the material being taught.

## Chapter Seven

## INTRODUCTION

The primary purpose of the present study was to examine the manner in which first and second grade students acquired concepts from the areas of number, numeration and the basic algorithms. Major variables used in the study were cognitive development, cognitive style, and language, each examined for its role in mathematics concept acquisition. The results of the study were obtained by following children throughout two years of their learning in the public school, with testing for the identified variables.

One stated purpose of the study was to develop hypotheses about the child's learning of mathematics concepts. Hypotheses were developed for the following research questions:

1. Are mathematics concepts presented to children before they are developmentally prepared for those concepts?
2. Does the teaching of mathematics favor a particular cognitive style?
3. What language should be used in teaching mathematics concepts to students whose native language is not English?

Each of these questions will be discussed in this chapter, with general research questions and/or hypotheses raised within the structure of each area. A short description of the findings leading to the hypothesis or research question will be given.

### Cognitive Development

The first question considers the issues raised by developmental psychology in the teaching of mathematics. Elsewhere in this report, the

statement has been made that while the mathematics curriculum is mathematically sound, it may not be psychologically sound. Consider the following questions:

1. Should mathematical concepts dependent upon the operation of "reversing" be delayed until the second grade?

Much of the data in this study suggest that certain mathematical operations were presented to children while they were in a stage of development where the concepts taught were too difficult for their reasoning capabilities. Objectives dealing with inverse operations and missing terms are dependent upon the operation of reversing; by definition, children at the preoperational stage do not have operations available to them. The researchers, in working with the children in the first grade, noted a great deal of difficulty with these types of objectives. Second graders, on the other hand, learned the missing addend concept with little difficulty. As a consequence, the following research hypotheses are offered:

Hypothesis #1: There will be no significant difference in achievement between first grade children taught objectives dependent upon reversals and students for whom those concepts are delayed until the second grade.

Hypothesis #2: Difficulties with mathematics concepts in the primary grades are significantly correlated with mathematics anxiety, as measured in later grades.

Hypothesis #3: Children for whom reversals are delayed until the second grade will learn those concepts more quickly and have less observed difficulty.

2. Should the use of two-digit numerals, especially as introduced through the use of expanded notation, be delayed until the second grade?

In addition to the fact that many of the first grade children were still operating at the preoperational or transitional stages, there is also evidence in the literature that a child can be at the concrete operational stage with smaller numerals, yet still be at the preoperational stage for larger numerals. Much of the evidence in this particular study supports that idea; children often performed a task successfully with small numerals, yet seemed overwhelmed by the same task when additional elements were placed in the set.

In addition to this, the "conservation of place value" task indicated that many children could not successfully relate grouped and ungrouped bundles of ten; consequently, teaching them the place value system for larger numerals seems questionable. Teachers also identified many expanded notation objectives as being difficult for their students. These objectives would be delayed if the child's introduction to larger numerals were presented in the second grade, rather than the first grade.

Hypothesis #1: If instruction in larger numerals is delayed until the second grade year, second graders will demonstrate significantly higher levels of achievement than will first grade children taught the same concepts.

Hypothesis #2: If instruction in larger numerals is delayed until the second grade year, children will demonstrate fewer difficulties with related place value concepts.

3. Is the spiral curriculum organizational pattern the best system to use for primary grade mathematics instruction?

The model for instruction presently used in mathematics is a traditional model for organizing instruction, called the "spiral approach." For example, division is first introduced to kindergarten children as the partitioning of a set; first graders then review the basic concept of separating sets into equal parts. Second graders are introduced to the basic facts of division, usually as the inverse of multiplication. Third graders practice the basic facts and are given timed tests over them. Fourth graders begin work on the algorithm for division. In other words, division is presented in a spiral form, with children at each age dealing with more difficult concepts.

Teachers identified the early, concept objectives as difficult; subsequent Piagetian task data indicated that children often did not possess the logical thinking structures necessary to perform the tasks required. As a result, a curricular organizational pattern which concentrated the early objectives nearer to the major instructional period might be preferable.

Hypothesis #1: A curricular organizational pattern which concentrates instruction will cause fewer difficulties than the "spiral" curriculum pattern.

4. Do children possess all the language structures which are utilized by teachers in the instructional process?

Language testing (in English) revealed that all students (both English- and Spanish-speaking) had some difficulties with the various language

structures used in the test situation. Some of the language structures tested involved concepts which were part of the math curriculum (the use of "biggest" or the use of mass versus count nouns), while some involved language structures used in the teaching of mathematics ("before" or "after," or the use of passive sentence construction forms). Test results indicated that at least some members of every group tested had difficulties with every structure tested.

As a result, two general research questions seem to be in order:

Question #1: What is the relationship between teacher language patterns and children's mathematics achievement?

Question #2: Are there specific language structures used in the teaching of mathematics which pose difficulties for students?

## Cognitive Style

It has been hypothesized in the literature that a contributing factor to the low achievement of Hispanic and Black students in mathematics is a "culturally influenced cognitive style, that is incompatible with the typical methods used in teaching mathematics" (Lysiak, 1978). From a review of the literature, there seems to be a need to study the relationship between cognitive style and mathematics achievement, and cognitive style, ethnicity and teaching methods.

In view of the results from the present study, the following research question seems in order:

1. Does the manner in which mathematics concepts and algorithms are presented to students favor a particular cognitive style?

Results obtained from tests administered to participating students indicated that a larger percent of field independent students scored above the 7th stanine in the Stanford Achievement Test (Math Comprehension and Computation), than field sensitive students. The largest difference between field sensitive and field independent students scoring above the 7th stanine was found in mathematics computation, when field independent students outperformed field sensitive students. In order to determine specifically where differences in performance existed, the Arithmetic Basic Skills Test was given to the students. Test scores indicated that field independent students tended to outperform field sensitive students in addition and subtraction concepts and computation. Both groups seemed to show similar performance in mastery of number and numeration concepts.

When these results were paired with observations made in the classroom, certain facts become evident. The way in which teachers taught number and numeration concepts seemed to be more appropriate for the cognitive styles of field sensitive students. The specific skills were presented in a global manner as opposed to a structural analytical subskills approach. Therefore, both groups performed similarly in number and numeration tasks. While teaching algorithms for addition and subtraction, teachers tended to teach a set of rules for the operations. This strategy may present a problem to the field sensitive students. Somehow, FS students could not follow the algorithm as presented by the teacher, while FI students did not seem to have any problems with this approach. An approach using "referents" (where "referents" refers to concrete objects familiar to the child; for example, using a set of marbles and cans to group the

marbles in tens, while teaching regrouping in the addition algorithm) might be more applicable to FS students.

In view of these findings, the following hypotheses would seem to be appropriate:

Hypothesis #1: There is no difference in computation between field independent and field sensitive students taught the addition algorithm by means of "referents."

Hypothesis #2: There is no difference in computation performance between field independent and field sensitive students taught the subtraction algorithm by means of "referents."

Precise Thinking Exercise

One component of the present study dealt with differences in performance between field sensitive and field independent students on algorithmic tasks, or Precise Thinking Exercises (PT). The objective was to observe the types of strategies used by the pupils in solving the tasks presented in order to develop hypotheses regarding the relationship between cognitive style and performance in mathematical algorithms. The following questions were asked:

1. Are the Precise Thinking (PT) Exercises, as originally designed for fifth-grade students, appropriate for first-grade students?

From the observational data gathered, all students were able to complete all of the exercises with the exception of the last activity, the "Complex Letter" exercise. This indicated that a modification to a format that may be more appropriate to students in the earlier grades was necessary.

Changes from symbolic to the pictorial format of the rules will be made, with less emphasis given on verbal instructions. Also, due to the reading problems shown by some of the students, a "symbolic to pictorial" modification is needed for the "Traffic Light" exercise. It was not clear whether the difficulties observed in this activity were due to the nature of the exercise, the low reading level of the students, or language problems on the part of limited English proficient students. This needs to be investigated further.

2. Are there any differences in performance on the PT exercises between field sensitive and field independent students?

Observations (recorded in Chapter Five) indicated a higher frequency of difficulties in the performance of the PT activities for the field sensitive students. This result may be explained in terms of an ability to shift psychological sets in thinking and the ability to reason inductively on literal and numerical patterns. The PT exercises seem to elicit these behaviors due to the structural patterns of the rules - the sequential stimulus/response design of the exercises and the hierarchical difficulty sequence of the activities. The observations seem to be in agreement with a study by Spitler (1971), who found that field independence is significantly related to the ability of shift psychological sets and the ability to make inductions from patterns.

3. What is the nature of the differences in performance that exist between the participating field sensitive and field independent students?

Most of the field sensitive students were observed to have difficulties with the differentiation of the "given" and "result" component of each rule. They took a longer period of time to identify the appropriate rule from the set of rules which corresponded to the pairing of exercise cards. Also, they were observed to have manipulation problems. It seemed as if the act of rule identification interfered with the replacement and selection of the appropriate "playing" cards. These results seem to be supported by a study reported by Witkin, et. al. (1978), in which a subject was asked to perform a task which required using traditional tools out of their conventional context. The study results showed that field sensitive subjects could not overcome the predominant context of the tools. In the PT exercises students were asked to take a set of rules from a pictorial context and apply them in a manipulative context in a series of sequentially structured tasks. The observations made in this study showed the field sensitive students having difficulties in making the shift between the two given contextual frames.

In view of the preceding discussion, the following hypothesis seems appropriate.

Hypothesis #1: Field sensitive students who participate in Precise Thinking Exercises will achieve higher in addition and subtraction than field sensitive students who do not perform the exercises.

Also, a research question dealing with an extension of the ideas presented may be posed:

Does instruction in abstract algorithmic methods facilitate the learning of algorithms in mathematics?

## Language

In dealing with the area of language, the present study focused on two types of students: the limited English proficient student, and the bilingual student (proficient in both languages). Concerning the limited English proficient student, the literature reports that in order for the students to acquire the basic concepts presented in the school curriculum, it is best to teach those concepts through the student's native language. This approach would eliminate the student having to translate from an unfamiliar language to his native language at the same time he/she is trying to understand a new concept.

Consider the third question posed at the beginning of this chapter:

> What language should be used in teaching mathematics concepts to students whose native language is other than English?

Results from this study indicate the Spanish monolingual and limited English proficient students had not mastered number, numeration and addition/subtraction concepts as well as bilingual and Hispanic English monolingual students. One obvious explanation for this poor performance is the fact that instruction was carried out completely in English, a language unfamiliar to the students. Also, teachers may have placed too heavy an emphasis on language in the explanation of those concepts. Spanish monolingual students also showed the lowest average performance of the three language groups in the addition and subtraction computation power and speed subtests of the Arithmetic Basic Skills Test. Again, the same

argument could be made to explain these results. The students cannot understand concepts presented in a language in which they are not proficient.

In view of these results, the following hypotheses may be stated:

Hypothesis #1: Spanish monolingual students taught number, numeration and addition and subtraction concepts in Spanish will outperform Spanish monolingual students taught those concepts in English.

Hypothesis #2: There is no significant difference in performance between Spanish monolingual students taught the addition and subtraction algorithms in Spanish and Spanish monolingual students taught in English.

Hypothesis #3: Limited English proficient students taught number, numeration and addition and subtraction concepts in Spanish will outperform limited English proficient students taught in English.

Hypothesis #4: There is no significant difference in performance between limited English proficient students taught the addition and subtraction algorithms in Spanish and English proficient students taught in English.

One of the areas in need of extensive research in bilingual education is that of classroom methodology. The basic question asked is "What are the appropriate methods to be used with limited English proficient students to that while they learn English, they can also acquire the concepts needed to function in the classroom?" One teaching approach, devised by

the Santa Cruz Materials Development Center at the University of Arizona, suggests a "concept preview strategy." Using this approach, the teacher, before presentation of a "main" lesson, conducts what is termed a "concept preview" lesson. The approach consists of two components. The first component deals with the teaching of the main concepts to be presented in the student's native language. The second component deals with the teaching of the English terminology for the concepts that will be presented in the main lesson. Once the students go through these two components, they acquire the concepts or skills needed through their native language and also become familiar with the English terminology, which will be presented in the main lesson. The main lesson, conducted in English, becomes a reinforcement of the concepts and terminology already presented. Although the approach seems a logical one, research is needed regarding the effectiveness of the strategies with limited English proficient students. In relation to this idea, the following hypothesis seems appropriate:

Hypothesis #1: Limited English proficient students taught through the "concept preview approach" will outperform limited English proficient students taught through a native language approach in -

a) understanding of concepts in the native language
b) understanding of the concepts in English
c) knowledge of terminology in English and/or native language for the areas of number, numeration and addition/subtraction concepts.

Concerning bilingual students, the literature indicates that youngsters who are bilingual have a wider experience in two cultures and languages, which give him/her advantages which a monolingual student does not enjoy. Peal and Lambert (1962) state that "intellectually, the experiences of bilingual students with two language systems seem to have left them with a mental flexibility, a superiority in concept formation, and a more diversified set of mental abilities than monolingual students." Results from the present study indicate that bilingual Hispanic students outperformed Spanish monolingual students and Hispanic English monolingual students in the areas of addition and subtraction concepts, addition and subtraction computation and in the mastery of number and numeration concepts. In view of this result, a final hypothesis seems to be in order:

Hypothesis #1: Bilingual students will outperform English monolingual students in mathematics achievement in the areas of number, numeration, addition/subtraction concepts and addition/subtraction computation.

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# APPENDIX A

CONTENT AREA: NUMBER: Cardinal Principle of Number

| Level | Number | Objective |
|---|---|---|
| LEVEL I | 11-71-02-01 | Identifies sets: more, fewer, same number of elements |
| | 11-71-01-04 | Orders sets, smallest number to largest number |
| | 11-11-01-01<br>11-11-01-02 | Identifies sets equal or unequal in number |
| | 11-71-01-01 | Identifies sets equal or unequal in number |
| | 11-71-01-03 | Makes second set equal in number to first set |
| | 11-11-01-05 | Makes non-equivalent sets equivalent |
| | 11-11-03-01<br>11-11-03-04 | Learner selects counters to show number of things, to ten |
| | 11-11-03-01a<br>11-11-03-04a | Learner selects numeral cards to show number of things to ten |
| LEVEL II | 11-71-02-01<br>11-71-02-02 | Identifies set that has more or fewer members |
| | 11-71-02-03 | Identifies set that has fewest, least, greatest, most members |
| | 11-11-01-03 | Identifies set having one more or one less, 0 - 5 |
| | 11-11-01-04 | Identifies set having one more or one less, 6 - 10 |
| | 11-11-01-06 | Establishes set equivalence |
| | 11-11-03-02 | Matches set with cardinal number, 0 - 5 |
| | 11-11-03-03 | Matches set with cardinal number, 6 - 10 |
| LEVEL VI | 11-71-03-02 | Uses $<$ or $>$ to describe order relation between pairs. |
| | 11-71-03-01 | Uses $=$ or $\neq$ in number sentences. |
| LEVEL VIII | 11-71-02-07 | Identifies larger or smaller number, 11- 100 |
| | 11-71-02-08 | Identifies smallest or greatest number in a triplet of numbers, 11 - 100 |
| | 11-71-03-03 | Uses $=$ or $\neq$ to describe related numbers |
| | 11-71-03-04 | Uses $<$ or $>$ to relate numbers 11 - 100 |

CONTENT AREA: NUMBER: Ordinal Principle of Number

| Level | Number | Objective |
|---|---|---|
| LEVEL I | 11-11-06-02 | Places picture in specific ordinal position |
| LEVEL III | 11-11-06-01 | Marks object in ordinal position, first to fifth |
| LEVEL V | 11-11-06-03 | Matches ordinals to object in positional order, first through fifth. |
| | 11-11-07-01<br>11-11-07-02<br>11-11-07-03 | Labels ordinal position of pictured events. |
| LEVEL VII | 11-11-06-04 | Matches ordinal words to given positions; 6th - 10th |
| LEVEL IX | 11-11-06-05 | Reads written ordinal words and matches to pictured ordered items in those positions, 11th to 20th |
| | 11-11-06-06 | Matches ordinal abbreviations to ordered pictured items, 1 - 19. |
| | 11-11-06-07 | Selects appropriate written ordinal word names for ordinal numbers, 25 - 100. |

CONTENT AREA: NUMERATION

| Level | Number | Objective |
|---|---|---|
| LEVEL I | 11-11-03-01a | Learner selects numeral cards to show number of things to 10. |
| | 11-31-01-03 | Identifies cardinal number of two sets, comparison. |
| LEVEL II | 11-71-02-04 | Identifies numeral that names greater number, 10 or less. |
| | 11-71-02-05 | Identifies numeral that names smaller number. |
| | 11-71-02-06 | Identifies numeral that names the largest number. |
| LEVEL III | 11-11-02-01<br>11-11-04-01 | Writes standard numeral. |
| | 11-11-02-02<br>11-11-04-02 | Writes standard numeral. |
| | 11-11-02-03<br>11-11-03-05 | Matches numeral to collection. |
| | 11-11-02-03<br>11-11-04-03 | Writes standard numeral. |
| | 11-11-05-01 | Writes standard numeral for written word names, 0 - 10. |
| LEVEL IV | 11-41-01-016 | Identifies N sets of M number of objects, N is 12 or less. |
| LEVEL V | 11-11-02-04<br>11-11-03-06<br>11-11-03-07 | Writes standard numeral from pictured collections, to 50. |
| | 11-11-09-02 | Selects standard numeral from picture, to 99. |
| | 11-11-08-01<br>11-11-08-04 | Counts by two's to 18, by ten's to 50. |
| | 11-11-09-01 | Writes expanded numeral from pictures, ten and ones. (to 99.) |
| | 11-11-10-01<br>11-11-11-01 | Writes standard numeral for standard expanded form. |
| | 11-11-12-01 | Writes expanded form for standard form. |

CONTENT AREA: NUMERATION

| Level | Number | Objective |
|---|---|---|
| LEVEL VII | 11-11-08-02<br>11-11-08-03 | Counts by three's to 18; by five's to 20. |
| | 11-11-08-05 | Gives cardinal number of sets grouped by 50's, 100's, for sets 200 to 100. |
| | 11-11-09-03 | Supplies expanded numeral from pictures, 100 - 999. |
| | 11-11-10-02 | Interprets positional value of specified digits, 3 digits. |
| | 11-11-11-02 | Selects or writes standard form or other expanded form for three-digit numeral. |
| LEVEL VIII | 11-71-02-01 | Identifies larger or smaller numeral, n to 100. |
| | 11-71-02-08 | Identifies smallest or greatest numeral in a triplet, 11 - 100 |
| LEVEL IX | 11-11-05-02 | Writes standard numerals for written word names; 11 - 99. |
| | 11-11-05-03<br>11-11-05-04 | Writes standard numerals for written word names, 100 - 1000. |
| LEVEL XI | 11-11-12-02 | Writes standard numeral for pictures, 1000 - 3000. |
| | 11-11-12-03 | Writes expanded form, 100 - 999. |
| | 11-11-05-06 | Matches numeral to written word name, 100 - 1000. |
| LEVEL XII | 41-32-01-01 | Identifies odd and even numerals, 1 - 100. |
| LEVEL XIII | 11-11-05-07<br>11-11-05-08 | Supplies numeral for written number, over 1000. |
| | 11-11-09-04<br>11-11-11-03 | Writes standard numeral or expanded form for written form. |
| | 11-11-12-04<br>11-11-02-05 | Writes standard numeral or expanded form for(a x 10) + b or (a x 0) + b where a and b are less than 10. |
| | 11-11-03-01 | Rounds 2-digit numerals to nearer ten. |
| | 41-32-01-01<br>41-32-01-02 | Classifies sums as odd or even. |

CONTENT AREA: NUMERATION

| Level | Number | Objective |
|---|---|---|
| LEVEL XIII (con't) | 41-32-01-03 | Classifies products as odd and even. |
| | 41-11-01-01 | Classifies number to 100 as prime, composites, or neither. |
| | 41-11-01-04 | Labels statements about primes and composites as T or F. |
| LEVEL XV | 11-11-10-04 | Gives value for specified digit, 4 - 6 digits. |
| | 11-11-11 04 | Standard numeral given for expanded form, 4 - 6 digits. |

CONTENT AREA: NUMERATION (Special Systems)

| Level | Number | Objective |
|---|---|---|
| LEVEL I | 31-21-01-01 | Identifies penny, nickel, dime. |
| LEVEL IV | 31-31-02-01<br>31-31-02-02 | Draws hour on clock. |
| | 31-31-02-03<br>31-31-02-04 | Draws hour hand on clock for half-past. |
| LEVEL VI | 31-21-01-02 | Identifies quarters and half dollars. |
| | 31-21-01-02 | Learner marks coins given in cents. |
| | 31-21-02-01 | Finds different ways to make nickel; dime. |
| | 31-21-02-02 | Finds different ways to make quarter. |
| | 31-31-02-05 | Makes clock show __:00 o'clock. |
| | 31-31-02-06 | Makes clock show __:30 o'clock. |
| | 31-31-03-01 | Arranges in order: days of the week. |
| LEVEL VIII | 31-21-01-04 | Matches pictures of coins to their values in cent notation: ≤ 25¢. |
| | 31-21-01-05 | Matches pictures of coins to written words: values to 25¢. |
| | 31-21-01-06 | Matches pictures of coins to value in dollar notation: ≤ 25¢. |
| | 31-21-01-07 | Writes values of pictured coins in both cent and dollar notation: ≤ 25¢. |
| | 31-31-03-02 | Arranges written names of the month in order. |
| | 31-31-03-03 | Identifies time lapsed as ___ months ___ weeks ___ days. |
| LEVEL X | 31-21-02-02 | Identifies combined equivalent to quarter. |
| LEVEL XI | 31-21-01-04 | Matches coins to values in ¢ (≤ 50¢). |
| | 31-21-01-05 | Matches coins to written word value half-dollar. |

CONTENT AREA: NUMERATION (Special Systems)

| Level | Number | Objective |
|---|---|---|
| LEVEL XI (con't) | 31-21-01-06 | Matches coins to values; dollar notation ($\leq 50$). |
| | 31-21-01-07 | Writes value of coins to 50 cents; cent and dollar notation. |
| LEVEL XII | 31-21-02-04 | Purchases; fewest coins. Total $\leq$ \$1.00. |

CONTENT AREA: ALGORITHMS (Addition)

| Level | Number | Objective |
|---|---|---|
| LEVEL I | 11-21-01-01<br>11-21-01-02 | Identifies cardinal number of 2 sets and cardinal number of union of 2 sets. |
| | 11-41-01-01 | Removes sets of specified cardinal number. |
| LEVEL III | 11-21-01-04<br>11-21-02-01<br>11-21-02-03 | Matches addition equation to pictured union sums to 10. |
| | 11-21-02-09 | Writes addition equation from number line, sums to 10. |
| | 11-21-02-11 | Writes equations to addition situations, sums to ten. |
| | 11-23-01-01<br>11-23-01-02 | Writes the sum of 2 addends, horizontal and vertical, sums to 10. |
| | 11-23-01-05 | Writes missing term (sum or addend), sums to 10. |
| | 11-31-01-03<br>11-31-01-07<br>11-31-01-08 | Translates number line representation into equations, sums to 10. |
| | 11-33-01-04 | Writes missing numerals (sums or addends), subtraction, sums to 10. |
| LEVEL IV | 11-23-01-07 | Horizontal form, sums to 10 one-minute timed test (20 problems). |
| | 11-23-01-08 | Vertical form, sums to 10 one-minute timed test (20 problems). |
| LEVEL V | 11-21-01-03<br>11-21-01-05<br>11-21-02-04 | Matches additional number sentences; pictorial representations used, sums 11 to 18. |
| | 11-21-02-05 | Writes number sentences; pictorial representations used, sums to 10. |
| | 11-21-02-06 | Writes number sentences; pictorial representations used, sums to 11 to 18. |
| | 11-21-02-07 | Writes number sentences from number line (sums to 11 to 18). |
| | 11-21-02-10 | Translates oral statements into an equation; sums to 18. |

CONTENT AREA: ALGORITHMS (Addition)

| Level | Number | Objective |
|---|---|---|
| LEVEL V (con't) | 11-21-02-11 | Identifies "plus," "addend," "sum," "equals," or "add." |
| | 11-21-02-12 | Groups addends by two's to find sums to 10. |
| | 11-21-02-13 | Groups addends by two's to find sums to 18. |
| | 11-23-01-03 | Finds sums (up to 4 addends) 1 digit addends (sums to 10). |
| | 11-23-01-04 | Completes addition matrix sums to 10. |
| | 11-23-01-06 | Finds missing terms (vertical form) (up to 4 addends) (sums to 10). |
| | 11-23-02-01 | Sums to 18; 2 addends horizontal form. |
| | 11-23-02-02 | Sums to 18; 2 addends vertical form. |
| LEVEL VI | 11-21-02-14 | Matches words to specific parts of addition equation (add, plus, equals, sums, addend). |
| | 11-24-01-03<br>11-24-02-01 | Matches addition word problem to equation (sums to 10). |
| | 11-24-02-02<br>11-24-03-01<br>11-24-03-04<br>11-24-02-03 | Writes addition equation for word problem to equation, sums to 10. |
| | 11-24-01-02<br>11-24-01-04<br>11-24-03-02<br>11-24-03-03 | Writes equation and solution for word problems, sums thru 18. |
| | 11-24-03-04 | Reads word problems and writes solution (sums to 10). |
| | 11-24-03-05 | Reads and writes solution sums to 18. |
| LEVEL VII | 11-22-01-01 | Matches examples of comm. prop. to number lines. |
| | 11-22-01-02 | Gives cardinal number of the union of sets as 10 plus ___, (sums 11 - 18). |

CONTENT AREA: ALGORITHMS (Addition)

| Level | Number | Objective |
|---|---|---|
| LEVEL VII (Con't) | 11-22-01-03 | Writes equations: form a + b = b + a using number lines. |
| | 11-22-01-04<br>11-22-01-05 | Complete addition equations, associative property, using number lines. |
| | 11-22-01-06 | Applies associative property (sums to 18). |
| | 11-23-02-03 | Completes addition matrices; sums 11 - 18. |
| | 11-23-02-04 | Addition equations; 2, 3, 4 addends missing terms; sums 11 - 18; horizontal. |
| | 11-23-02-05 | Addition equations; 2, 3, 4 addends missing terms; sums 11 - 18; vertical. |
| | 11-23-02-06 | One minute timed test; horizontal, sums 11 - 18. |
| | 11-23-02-07 | One minute timed test; vertical, sums 11 - 18. |
| | 11-23-02-08 | Vertical form; 3 - 4 - 5 digit addends sums to 18. |
| | 11-23-03-01 | Sums, 1 and 2 - digit addends; horizontal and vertical; no regrouping. |
| | 11-23-03-02 | Sums, 1 and 2 - digit addends; horizontal and vertical; regrouping. |
| | 11-23-04-01 | Sums, 2 or 3 digit addends, vertical; no regrouping. |
| LEVEL VIII | 11-24-02-02 | Writes equation for word problems, sums through 18. |
| | 11-24-03-02 | Solves word problems; sums 11 - 18. |
| | 11-62-01-04 | Selects addition equations from subtraction equations. Sums $\leq 10$. |
| | 11-62-01-06 | Selects addition equations from subtraction equations; sums 11 - 18. |
| | 31-23-02-01 | Writes sums in cent notation: sums $\leq$ 10¢. |
| | 31-23-02-02 | Writes sums in cent notation: sums $\leq$ 20¢. |
| | 31-24-01-02a | Writes total value of pictured items labeled as to value; value $\leq$ 10¢. |
| | 31-24-01-02 | Finds value of selected items and amount of change. Value $\leq$ 10¢ |

CONTENT AREA: ALGORITHMS (Addition)

| Level | Number | Objective |
|---|---|---|
| LEVEL IX | 11-21-02-08 | Writes addition equations from verbal descriptions of additive situations. |
| | 11-31-01-04<br>11-31-01-05 | Determines number by which one set has more or fewer items and selects appropriate sentence. |
| LEVEL X | 11-62-01-07 | Matches addition equation to subtraction equation. Sums ≤ 100. |
| | 31-24-01-01 | Writes total value of labeled items. Value ≤ 25¢. |
| | 31-24-01-03 | Selects items; finds total value. Writes change. Values ≤ 25¢. |
| LEVEL XI | 11-23-04-02 | Sums, 2 or 3 digit addends, vertical; regrouping. |
| | 11-24-01-05<br>11-24-01-06 | Matches word problem to addition equation. Sums 100. |
| | 11-24-02-03 | Writes equation for addition word problems; sums 100. |
| | 11-24-03-06<br>11-24-03-07 | Writes solutions for addition word problems sums (100 - 1000). |
| | 11-24-03-08 | Reads and solves addition word problems, sums ≤ 1000. |
| LEVEL XII | 31-23-01-04 | Adds money, dollars and cents, horizontal and vertical. |
| LEVEL XIII | 11-23-02-08<br>11-23-02-11 | Missing terms, 2 - 3 digit addition regrouping. |
| LEVEL XV | 11-23-04-03 | Finds sums, vertical form, six or less digits. |

CONTENT AREA: ALGORITHMS (Subtraction)

| Level | Number | Objective |
|---|---|---|
| LEVEL I | 11-31-01-01<br>11-31-01-02 | Separates sets, identifies cardinal number of subsets. |
| LEVEL III | 11-31-01-03<br>11-31-01-07<br>11-31-01-10 | Matches number line to subtraction sentence (sums to 10). |
| | 11-31-01-03<br>11-31-01-07<br>11-31-01-09 | Matches picture to related subtraction sentence (take away). |
| | 11-31-01-03<br>11-31-01-07<br>11-31-01-09 | Matches picture to related subtraction sentence (comparative). |
| | 11-33-01-01<br>11-33-01-02<br>11-33-01-03 | Writes differences, vertical and horizontal form, sums to 10. |
| LEVEL IV | 11-33-01-05 | Horizontal forms, differences to 10, one-minute timed tests (20 problems). |
| | 11-33-01-06 | Vertical forms, differences to 10, one-minute timed tests (20 problems). |
| LEVEL V | 11-31-02-01 | Identifies "minus," "equal," "subtract," "minuend," "subtrahend," "difference," "sum," "addend." |
| | 11-31-01-03b | Horizontal form, sums to 18, supplies differences. |
| | 11-31-01-03b<br>11-33-02-02 | Vertical form, sums to 18, supplies the differences. |
| | 11-31-01-03b<br>11-32-02-04 | Writes missing terms, horizontal and vertical forms - - sums 11 to 18. |

CONTENT AREA: ALGORITHMS (Subtraction)

| Level | Number | Objective |
|---|---|---|
| LEVEL VI | 11-34-01-01<br>11-34-01-02<br>11-34-01-03<br>11-34-02-01 | Translates problems into subtraction story problems. (Sums thru 10). |
| | 11-34-01-04 | Matches story problem to corresponding equations (sums thru 18) |
| | 11-34-02-02<br>11-34-02-03 | Matches subtraction word problem to equation (sums 11 to 18). |
| LEVEL VII | 11-32-01-01 | Subtraction as non-commutative operation number lines. |
| | 11-33-02-05 | Subtraction; one minute timed test horizontal sums 11 - 18. |
| | 11-33-02-06 | One minute timed test, subtraction; vertical, sums 11 - 18. |
| | 11-33-03-01 | Supplies differences 1 and 2 digit; vertical; no regrouping. |
| | 11-33-03-02 | Supplies differences 2 and 3 digit numerals; no regrouping. |
| | 11-33-03-03 | Supplies missing terms, subtraction; 1 and 2 digit; no regrouping. |
| | 11-33-03-04 | Supplies missing terms, subtraction; 2 and 3 digit; horizontal, no regrouping. |
| | 11-33-03-05 | Supplies missing terms, subtraction; up to 3 digits; norizontal and vertical; no regrouping. |
| LEVEL VIII | 11-62-01-01<br>11-62-01-02 | Selects subtraction equations from addition equations. Sums $\leq$ 10. |
| | 11-62-01-05 | Selects subtraction equations from addition equations. Sums 11 - 18. |
| | 31-33-02-01 | Writes differences in cent notation: sums $\leq$ 10¢. |
| | 31-33-02-02 | Writes differences in cent notation: sums $\leq$ 20¢. |

CONTENT AREA: ALGORITHMS (Subtraction)

| Level | Number | Objective |
|---|---|---|
| LEVEL IX | 11-31-01-04<br>11-31-01-05 | Determines number by which one set has more or fewer items and selects appropriate number sentences. |
| | 11-31-01-06 | Supplies or selects number sentences for pictured set subtraction scenes. |
| | 11-34-01-05 | Writes word problems for pictured subtraction situations. |
| | 11-34-01-06<br>11-34-01-07<br>11-34-01-08 | Selects subtraction story problems for $a - b = c$, $a \leq 18$. |
| | 11-34-03-01 | Reads written subtraction word problems with examiner and solves; sums to 10. |
| | 11-34-03-02 | Reads written subtraction word problems with examiner and solves; sums 11 - 18. |
| LEVEL X | 11-62-01-07 | Matches subtraction equation to addition equation. Sums $\leq$ 100. |
| | 31-24-01-03 | Selects items; finds total value. Writes change. Values $\leq$ 25¢. |
| LEVEL XI | 11-31-02-01<br>11-31-02-02 | Selects best minuend, regrouping tens to ones; hundreds to tens; 2 and 3 digits. |
| | 11-31-02-03<br>11-31-02-05 | States whether regrouping is needed, 2 and 3 digit subtraction problems. |
| | 11-32-01-03 | Relates zero concepts to subtraction problems. |
| | 11-34-02-03<br>11-34-02-04<br>11-34-02-05<br>11-34-02-07 | Selects equations for subtraction word problems; 2 and 3 digits; no regrouping. |
| | 11-34-02-06 | Equations which are not translations of subtraction word problems, 2 and 3 digits, no regrouping. |
| | 11-34-03-03 | Solves subtraction word problem sums (18 - 99) no regrouping. |

CONTENT AREA: ALGORITHMS (Subtraction)

| Level | Number | Objective |
|---|---|---|
| LEVEL XII | 31-23-01-03 | Subtracts money cent notation, no regrouping, sums ≤ $1.00. |
| | 31-23-02-05 | Subtracts money, dollar and cents, no regrouping, sums ≤ $10.00. |
| LEVEL XIII | 11-33-03-02<br>11-33-03-03<br>11-33-03-04 | Missing terms, 2 - 3 digit subtraction. |
| | 11-33-03-05 | Finds differences, 2 - 3 digit subtraction, no regrouping. |
| | 11-33-03-06<br>11-33-03-07 | Finds differences, 1, 2, 3 digit subtraction. Regrouping. |
| | 11-33-03-07<br>11-33-03-08 | Missing terms, 2 - 3 digit subtraction. Regrouping. |
| | 11-34-03-07 | Subtraction word problems, sums 18 to 99, regrouping. |
| | 31-23-02-04 | Subtracts money, cent notation, to 99¢ with regrouping. |
| LEVEL XV | 11-33-04-01<br>11-33-04-02 | Finds differences (4 - 6 digit) horizontal and vertical without regrouping. |
| | 11-33-04-03 | Supplies missing terms, vertical subtraction without regrouping. |
| | 11-33-04-04<br>11-33-04-05 | Subtraction (4 - 6 digit) horizontal and vertical, with regrouping. |
| | 11-33-04-06 | Supplies missing terms (4 - 6 digit) horizontal with regrouping. |

CONTENT AREA: ALGORITHMS (Multiplication)

| Level | Number | Objective |
|---|---|---|
| LEVEL VIII | 11-41-01-01 | Interprets multiplication; identifies factor set, product set, and completes equations. Products ≤ 25. |
| | 11-44-01-01<br>11-44-01-04 | Translates word problems into equations and solves products ≤ 25. |
| LEVEL IX | 11-41-01-02 | Supplies expressions from pictured set situations; a x b, products ≤ 25. |
| | 11-41-01-04 | Interprets expressions, horizontal form, $a_1 + a_2 + a_3 + \ldots + a_n = n$ sets of a or $n \cdot a$, product ≤ 25. |
| | 11-41-01-05 | Interprets expressions, vertical format $a_1 + a_2 + \ldots + a_n = n$ sets of a or $n \cdot a$, products ≤ 25. |
| | 11-41-01-06<br>11-41-01-07 | Writes multiplication and addition number sentences; products and sums ≤ 25. |
| | 11-41-01-08 | Supplies number names of equal addends and related factors from number lines showing repeated addition. |
| | 11-41-01-09<br>11-41-01-10 | Supplies multiplication sentences from arrays. |
| | 11-41-04-01 | Uses multiplication to name the number of possible pairings between elements of two sets # of pairings ≤ 25. |
| | 11-41-04-02 | Organizes pairings of 2 sets to show multiplication situations ≤ 25. |
| | 11-42-01-01 | Identifies the multiplication properties of 1 and 0 by writing products or completing generalizations concerning the effect 1 and 0 have upon products. |
| | 11-42-01-02<br>11-42-01-03 | Uses commutative and associative properties to complete multiplication number sentence and matrix. |
| | 11-43-01-01<br>11-43-01-02 | Writes products in horizontal and vertical form, products to 25. |
| | 11-43-01-03 | Completes multiplication sentences and matrix products to 25. |
| | 11-43-01-04 | Two-minute timed test, products to 25, vertical form. |

CONTENT AREA: ALGORITHMS (Multiplication)

| Level | Number | Objective |
|---|---|---|
| LEVEL X | 11-44-01-01<br>11-44-01-05 | Multiplication word problems equations and solutions: products ≤ 25. |
| | 11-62-02-03 | Matches multiplication equations to division equations: products ≤ 25. |
| LEVEL XI | 11-41-01-01 | Multiplication facts from pictures; products 26 - 45. |
| | 11-41-01-02<br>11-41-01-03 | Product expressions from pictures 26 - 45. |
| | 11-41-01-04/05/06/07 | Addition and multiplication equations from number lines (26 - 45). |
| | 11-41-01-09<br>11-41-01-10 | Multiplication equations from rectangular arrays (26 - 45). |
| | 11-41-05-01 | Matches words to symbols in addition and multiplication equations (26 - 45). |
| | 11-42-01-01 | Multiplication properties of 1 and 0 (products 26 - 45). |
| | 11-42-01-02<br>11-42-01-03<br>11-43-02-01 | Complete multiplication matrix products to 45. |
| | 11-43-02-02<br>11-43-02-03 | Products through 25, horizontal and vertical. |
| | 11-43-06-01 | Matches number to its multiples to 45. |
| | 11-44-01-02 | Writes equation and solves multiplication word problems: products to 45. |
| LEVEL XII | 11-43-02-04 | Products; 1 minute timed test; products ≤ 45. |
| | 31-43-04-01 | Multiplies money by whole number; dollar notation, product ≤ 45 |
| | 31-43-04-02 | Multiplies money by whole number; dollar notation, product ≤ $.45. |

CONTENT AREA: ALGORITHMS (Multiplication)

| Level | Number | Objective |
|---|---|---|
| LEVEL XIII | 11-41-01-01/02/03 | Matches expressions to picture products. |
| | 11-41-01-04 | Matches products to repeated addition products 46 - 81. |
| | 11-41-01-06<br>11-41-01-07 | Writes addition and multiplication equations from picture products 46 - 81. |
| | 11-41-01-08 | Number lines, repeated addition and multiplication. Products 46 - 81. |
| | 11-41-01-09<br>11-41-01-10 | Writes multiplication equations from arrays. Products 46 - 81. |
| | 11-43-01-05 | Missing terms from multiplication facts to 81. |
| | 11-43-03-01<br>11-43-03-02 | Writes products to 81. |
| | 11-43-06-03 | Writes multiplication expressions for multiples of N (46 - 81). |
| | 11-43-06-04<br>11-43-06-05 | Names multiples of N; Identifies N from a set of multiples of N. |
| | 11-44-01-03 | Writes equations and solves word problems, products to 81. |
| LEVEL XIV | 11-43-03-04 | Timed test 36 products to 81 in one minute. |
| LEVEL XV | 11-43-04-03<br>11-43-04-04 | Finds products: 1 digit number, x any whole number. |
| | 11-45-01-01 | Selects missing digit; multiply 1 digit x any whole number. |

CONTENT AREA: ALGORITHMS (Division)

| Level | Number | Objective |
|---|---|---|
| LEVEL I | 11-51-01-01 | Partitions a set when given specified number in set. |
| | 11-51-01-02 | Partitions set when given specified number of subsets. |
| LEVEL VI | 11-41-01-01 | Identifies N sets with M objects, (total ≤ 25). |
| | 11-51-01-01<br>11-51-01-03 | Uses pictures to interpret division (total ≤ 25). |
| | 11-51-01-04 | Partitions set into N subsets of N with M elements (total ≤ 25). |
| | 11-51-01-05 | Partitions sets with number line drawings (total ≤ 25). |
| LEVEL X | 11-51-01-01/02/03 | Given equivalent subsets, reports ___ subsets of ___ with ___ remaining. |
| | 11-51-01-04 | Forms equivalent subsets; reports ___ subsets of ___ with ___ remaining. |
| | 11-51-01-05 | Uses number line to report: N = ____ sets of ____. |
| | 11-51-01-06 | Matches repeated subtraction to division products ≤ 25. |
| | 11-51-02-01a<br>11-51-02-02a<br>11-53-02-01 | Writes missing factor; form a ÷ b. Products ≤ 25. |
| | 11-53-02-02 | Writes missing _____ factor; form $b\overline{)a}$. Products ≤ 25. |
| | 11-53-02-03 | Timed test; missing factor; a ÷ b or $b\overline{)a}$. Products ≤ 25. |
| | 11-52-01-01 | Division properties of 0 and 1. |
| | 11-53-02-04 | Writes missing terms; form a ÷ b. Products ≤ 25. |
| | 11-53-02-05 | Writes missing terms; form $b\overline{)a}$. Products ≤ 25. |
| | 11-53-02-06 | Timed test; missing terms; a ÷ b or $b\overline{)a}$. Products ≤ 25. |
| | 11-54-01-01 | Writes equations and solutions; division word problems. Products ≤25. |
| | 11-62-02-01<br>11-62-02-02 | Matches division equations to multiplication equations. Products ≤ 25. |

CONTENT AREA: ALGORITHMS (Division)

| Level | Number | Objective |
|---|---|---|
| LEVEL XII | 11-51-01-01/02/03 | Express partitioning of sets (26-45) as: N = ____ subsets of ____ with ____ remaining. |
| | 11-51-01-04 | Partition sets (26-45) as: N = ____ subsets of ____ with ____ remaining. |
| | 11-51-01-05 | Partitioning line segments: (26-45) parts; complete statements |
| | 11-51-01-06 | Matches repeated subtraction problems to division statements (26-45). |
| | 11-53-02-01<br>11-53-02-02 | Writes missing factor for division. Products (26-45). |
| | 11-53-02-04<br>11-53-02-05 | Writes missing numeral, division: (26-45). |
| | 11-54-01-02 | Writes equation and solves division word problems (26-45). |
| | 11-62-02-04<br>11-62-02-05 | Matches multiplication equations to related division equations and vice versa. |
| | 31-53-04-01a | Divides money by whole number; cent notation, products ≤ 45¢. |
| | 31-53-04-01b | Divides money by whole number; dollar notation, products ≤ $.45 |
| LEVEL XIII | 11-53-02-03<br>11-53-02-06 | Missing terms; division products 26-45, 1 minuted timed. |
| | 11-51-01-01/02/03 | Writes ___ subsets ___ with (remaining) = N. from picture. (products 45-81). |
| | 11-51-01-04 | Partition sets and writes ___ subsets of ___ with ___ (remainin = N. (products 45-81). |
| | 11-51-01-05 | Writes ___ jumps of ___ with ___ remaining = N. from number line (46-81). |
| | 11-51-01-06 | Matches repeated subtraction to division; products. (46 - 81). |
| | 11-51-02-01<br>11-51-02-02 | Writes forms for division $\sqrt{}$, $\div$, etc. |
| | 11-53-02-01/02/03/04/05 | Writes missing term, division, products. (46 - 81). |

CONTENT AREA: ALGORITHMS (Division)

| Level | Number | Objective |
|---|---|---|
| LEVEL XIV | 11-53-02-03 | Timed test. Supplies missing factors for division (46 - 81) one minute. |
| | 11-53-02-06 | Timed test. Supplies missing terms, 15 division exercises (46 - 81) one minute. |
| LEVEL XV | 11-53-04-01 | Divides: 3-digit dividend, 1 digit divisor, zero remainder. |
| | 11-53-04-02 | Divides: 3-digit dividend, 1 digit divisor, one rename, zero remainder. |
| | 11-53-04-03 | Divides any number by 1 digit divisor; zero remainder, zero(s) in dividend. |
| | 11-53-04-04<br>11-53-04-05 | Divides any number by 1 digit divisor; non-zero remainder. |
| | 11-54-01-03 | Solves division word problems products $\leq 81$. |

APPENDIX B

Table One

Summary of Significant $X^2$ Tests for
Teachers of Different Linguistic Groups
With Responses to Objectives

| # | Level | Objective | df | $X^2$ | p | Cramer's V |
|---|---|---|---|---|---|---|
| 1. | I | Identifies sets: more, fewer, same number of elements. | 4 | 10.636 | .05 | .63960 |
| 2. | II | Establishes set equivalence. | 3 | 7.809 | .05 | .47234 |
| 3. | II | Matches set with cardinal number 0 to 5. | 3 | 12.939 | .01 | .60803 |
| 4. | II | Matches set with cardinal number 6 to 10. | 3 | 13.052 | .01 | .61068 |
| 5. | III | Writes standard numeral (0-5). | 2 | 15.980 | .001 | .54399 |
| 6. | III | Writes standard numeral (6 - 10.) | 3 | 8.772 | .05 | .40306 |
| 7. | III | Matches numeral to collection (11 - 20.) | 3 | 9.833 | .05 | .43704 |
| 8. | V | Labels ordinal position of pictured events. | 4 | 12.089 | .05 | .49670 |
| 9. | V | Writes number sentences, pictoral representations used, sums to 10. | 4 | 10.759 | .05 | .47344 |
| 10. | V | Writes number sentences, pictorial representations used, sums 11 to 18. | 4 | 11.798 | .05 | .49578 |
| 11. | V | Writes number senteces from number line (sums 11 to 18.) | 3 | 7.855 | .05 | .40453 |
| 12. | VI | Matches words to specific parts of addition equation (add, plus, equals, sums addend) | 4 | 11.531 | .05 | .49531 |
| 13. | VI | Partitions set into N subsets of N with M elements (total $\leq$ 25.) | 4 | 9.490 | .05 | .45421 |

Table One (Con't)

Summary of Significant $X^2$ Tests for
Teachers of Different Linguistic Groups
With Responses to Objectives

| # | Level | Objective | df | $X^2$ | p | Cramer's V |
|---|---|---|---|---|---|---|
| 14. | VII | Interprets positional value of specified digits, 3 digit numerals. | 3 | 10.989 | .05 | .48876 |
| 15. | VII | Writes equations: form a + b = b + a, using number lines. | 3 | 9.462 | .05 | .45353 |
| 16. | IX | Supplies expressions from pictured set situations; a x b, products $\leq$ 25. | 3 | 8.416 | .05 | .59216 |
| 17. | IX | Organizes pairings of 2 sets to show multiplicative situations $\leq$ 25. | 2 | 7.272 | .05 | .55048 |
| 18. | IX | Writes products in horizontal and vertical form, products to 25. | 3 | 9.143 | .05 | .61721 |
| 19. | X | Forms equivalent subsets. Reports __ subsets of __ with __ remaining. | 2 | 8.558 | .05 | .56300 |
| 20. | XI | Matches number to its multiples, to 5. | 3 | 9.125 | .05 | .69302 |
| 21. | XI | Writes equation and solves multiplication word problems, products to 45. | 3 | 7.924 | .05 | .64579 |
| 22. | XII | Partitioning line segments: (26 to 45). | 2 | 5.805 | .05 | .52578 |
| 23. | XII | Matches repeated subtraction problems to division statements (26 to 45.) | 3 | 7.699 | .05 | .60513 |
| 24. | XII | Subtracts money, cent notation, no regouping, sums $\leq$ $1.00. | 3 | 8.182 | .05 | .63960 |

Table One (Con't)

Summary of Significant $X^2$ Tests for
Teachers of Different Linguistic Groups
With Responses to Objectives

| # | Level | Objective | df | $X^2$ | p | Cramer's V |
|---|---|---|---|---|---|---|
| 25. | XII | Multiplies money by whole numbers; cent notation; products $\leq$ 45¢. | 4 | 12.857 | .05 | .80178 |
| 26. | XIII | Writes standard numeral or expanded form for (a x 10) + b or (a x 0) + b, a$\leq$9, b$\leq$9. | 4 | 10.036 | .05 | .64667 |
| 27. | XIII | Finds difference, 2-3 digit subtraction, no regouping. | 4 | 10.469 | .05 | .67468 |
| 28 | XIII | Finds difference, 1, 2, 3, digit subtraction, regrouping. | 4 | 15.541 | .01 | .80471 |
| 29. | XIII | Number lines, repeated addition and multiplication, products 46-81. | 3 | 11.031 | .01 | .67796 |
| 30. | XIII | Missing terms from multiplication facts to 81. | 3 | 8.245 | .05 | .59872 |
| 31. | XIII | Writes products to 81. | 2 | 17.500 | .001 | .91287 |
| 32. | XIII | Missing terms; division products 26-46, one minute, timed. | 3 | 11.943 | .01 | .73679 |
| 33. | XIII | Writes missing term, division products, 46 - 81. | 3 | 8.523 | .05 | .63707 |
| 34. | XIII | Subtracts money, cent notation to 99¢ with regrouping. | 3 | 8.267 | .05 | .61302 |
| 35. | XIII | Labels statement about primes and composites as T or F. | 2 | 7.570 | .05 | .60041 |
| 36. | XV | Supplies missing terms, vertical subtraction without regrouping. | 4 | 11.633 | .05 | .78246 |